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CONTENTS

Mathematics. — *Intuitionistische Betrachtungen über den Formalismus* [1]).
By Prof. L. E. J. BROUWER.

(Communicated at the meeting of December 17, 1927).

LITERATUR.

HILBERT.

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[1]) In ungefähr gleichlautender Form am 16.2.1928 der Berliner Akademie der Wissenschaften vorgelegt.

§ 1.

Die Richtigkeitsdifferenzen zwischen der formalistischen Neubegründung und dem intuitionistischen Neubau der Mathematik werden beseitigt sein, und die Wahl zwischen beiden Beschäftigungen sich auf eine Geschmacksangelegenheit reduzieren, sobald die folgenden in erster Linie auf den Formalismus bezüglichen, aber in der intuitionistischen Literatur zuerst formulierten Einsichten allgemein durchgedrungen sein werden. Dieses Durchdringen ist deshalb nur eine Zeitfrage, weil es sich um reine Besinnungsergebnisse handelt, die kein diskutables Element enthalten und zu denen jederman der sie einmal verstanden hat, sich bekennen muss. Von den vier Einsichten ist bisher für zwei dieses Verständnis und dieses Bekenntnis in der formalistischen Literatur erreicht. Das Eintreten der gleichen Sachlage für die beiden übrigen wird das Ende des Grundlagenstreites in der Mathematik bedeuten.

Erste Einsicht. *Die Einteilung der formalistischen Bemühungen in einen Aufbau des „mathematischen Formelbestandes" (formalistischen Bildes der Mathematik) und eine intuitive (inhaltliche) Theorie der Gesetze dieses Aufbaues, sowie die Erkenntnis, dass für die letztere Theorie die intuitionistische Mathematik der Menge der natürlichen Zahlen unentbehrlich ist.*

Zweite Einsicht. *Die Verwerfung der gedankenlosen Anwendung des logischen Satzes vom ausgeschlossenen Dritten, sowie die Erkenntnis, erstens dass die Erforschung des Berechtigungsgrundes und des Gültigkeitsbereichs des genannten Satzes einen wesentlichen Gegenstand der mathematischen Grundlagenforschung ausmacht, zweitens dass dieser Gültigkeitsbereich in der intuitiven (inhaltlichen) Mathematik nur die endlichen Systeme umfasst.*

Dritte Einsicht. *Die Identifizierung des Satzes vom ausgeschlossenen Dritten mit dem Prinzip von der Lösbarkeit jedes mathematischen Problems.*

Vierte Einsicht. *Die Erkenntnis dass die (inhaltliche) Rechtfertigung der formalistischen Mathematik durch den Beweis ihrer Widerspruchslosigkeit einen circulus vitiosus enthält, weil diese Rechtfertigung auf der (inhaltlichen) Richtigkeit der Aussage, dass aus der Widerspruchslosigkeit eines Satzes die Richtigkeit dieses Satzes folge, d. h. auf der (inhaltlichen) Richtigkeit des Satzes vom ausgeschlossenen Dritten beruht.*

1. Die erste Einsicht fehlt noch in F 2 (vgl. insbesondere den mit derselben in Widerspruch stehenden Absatz V auf S. 184—185). Nachdem sie durch POINCARÉ stark vorbereitet worden war, tritt sie zum ersten Mal in der Literatur auf in I 1, wo S. 173—174 die genannten Teile der formalistischen Mathematik als *mathematische Sprache* und *Mathematik 2. Ordnung* unterschieden und der intuitive Charakter des letzteren Teiles betont wird[2]). Mit der Bezeichnung der Mathematik

2) Eine mündliche Erörterung der ersten Einsicht Herrn HILBERT gegenüber hat im Herbst 1909 in mehreren Unterhaltungen stattgefunden.

2. Ordnung als *Metamathematik* ist sie in F 4 (vgl. insbesondere S. 165 u. 174) in der formalistischen Literatur durchgebrochen. Der Anspruch der formalistischen Schule, mit dieser dem Intuitionismus entnommenen Einsicht den Intuitionismus ad absurdum zu führen (vgl. Math. Zeitschr. 26, S. 3), ist wohl nicht ernst zu nehmen.

2. Die gedankenlose Anwendung des logischen Satzes vom ausgeschlossenen Dritten findet sich noch in F 2 und F 3 (vgl. z. B. F 3, S. 413, Z. 11—4 v. u., und insbesondere F 2, S. 182, Z. 16—19 v. o., S. 182, Z. 2 v. u. —S. 183, Z. 2 v. o., S. 184, Z. 21—13 v. u., wo jedesmal der Satz vom ausgeschlossenen Dritten als mit dem Satz vom Widerspruch im wesentlichen gleichbedeutend angesehen wird). Zum ersten Male findet sich die zweite Einsicht in der Literatur in I 2, und sodann mehr oder weniger ausführlich in jeder der Veröffentlichungen I 3—8. Abgesehen von der mit ihr aufs engste verbundenen Erkenntnis der intuitionistischen Widerspruchslosigkeit des Satzes vom ausgeschlossenen Dritten, bricht sie in der formalistischen Literatur durch in F 5 [3]), wo einerseits die beschränkte inhaltliche Gültigkeit des Satzes vom ausgeschlossenen Dritten anerkannt (vgl. insbesondere S. 155—156), andererseits die widerspruchslose Kombination einer logischen Formulierung des Satzes vom ausgeschlossenen Dritten mit anderen Axiomen im Rahmen der formalistischen Mathematik als Aufgabe gestellt wird. Besonders eloquent wird dann auf die beschränkte inhaltliche Gültigkeit des Satzes vom ausgeschlossenen Dritten hingewiesen in F 6, S. 173—174, wo aber die Erweiterung seiner Anzweiflung auf die übrigen Aristotelischen Gesetze über das Ziel hinausschiesst.

3. Während der Zeit der gedankenlosen Anwendung des Satzes vom ausgeschlossenen Dritten in der formalistischen Literatur wird das Prinzip von der Lösbarkeit jedes mathematischen Problems zunächst in F 1, S. 52 als Axiom bzw. Ueberzeugung, sodann in F 3, S. 412—413 in zwei verschiedenen Formen (in welchen statt von „Lösbarkeit" der Reihe nach von „prinzipieller Lösbarkeit" und von „Entscheidbarkeit durch eine endliche Anzahl von Operationen" gesprochen wird) als Gegenstand noch zu erledigender Probleme hingestellt. Aber auch nach der Erörterung der dritten Einsicht in I 2, S. 156, I 4, S. *80*, I 6, S. 203—204, und nach dem Durchbruch der zweiten Einsicht in der formalistischen Literatur wird in F 6, S. 180, wo das Problem der Widerspruchsfreiheit des Axioms von der Lösbarkeit eines beliebigen mathematischen Problems als Beispiel einer „in den mathematischen Denkbereich fallenden Frage grundsätzlicher Art, an die man sich früher nicht heranmachen konnte" hingestellt wird, diese Frage als unabhängig von der Sicherung der (die Widerspruchsfreiheit des Satzes vom ausgeschlossenen Dritten mit umfassenden) Grundlagen der mathematischen Wissenschaft noch offenstehend vorgeführt.

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[3]) Nachdem schon in F 4, S. 160 Aufmerksamkeit auf den Satz vom ausgeschlossenen Dritten bekundet wird.

4. Die vierte Einsicht wird zum Ausdruck gebracht in I 9, S. 64. In der formalistischen Literatur findet sich von ihr bisher keine Spur, wohl aber manche ihr widersprechende Aeusserung, z.B. in F 1, S. 55—56 und vor allem in F 6, wo S. 162—163 noch ausgerufen wird: „Nein, wenn über den Nachweis der Widerspruchsfreiheit hinaus noch die Frage der Berechtigung zu einer Massnahme einen Sinn haben soll, so ist es doch nur die, ob die Massnahme von einem entsprechenden Erfolge begleitet wird" [4]).

Nach dem Vorstehenden hat der Formalismus vom Intuitionismus nur Wohltaten empfangen und weitere Wohltaten zu erwarten. Dementsprechend sollte die formalistische Schule dem Intuitionismus einige Anerkennung zollen, statt gegen denselben in höhnischem Ton zu polemisieren und dabei nicht einmal die richtige Erwähnung der Autorschaft einzuhalten. Ueberdies sollte die formalistische Schule bedenken, dass im Rahmen des Formalismus von der eigentlichen Mathematik bisher noch immer *nichts* gesichert ist (weil ja der metamathematische Widerspruchsfreiheitsbeweis des Axiomensystems nach wie vor aussteht), wogegen der Intuitionismus auf der Grundlage seiner konstruktiven Mengendefinition und seiner Haupteigenschaft der finiten Mengen [5]) schon einige Lehrgebäude der eigentlichen Mathematik in unerschütterlicher Sicherheit neu errichtet hat. Wenn also die formalistische Schule nach ihrer Aeusserung in F 6, S. 180 beim Intuitionismus Bescheidenheit bemerkt hat, so sollte sie darin Anlass finden, in bezug auf diese Tugend dem Intuitionismus nicht nachzustehen.

§ 2.

In I 7, S. 3 wurde bemerkt, dass bei den Bestrebungen zur Durchführung des Widerspruchsfreiheitsbeweises der formalistischen Metamathematik die intuitionistische Widerspruchslosigkeit des Satzes vom ausgeschlossenen Dritten als ermutigender Umstand gelten kann.

Wenn die kombinierte Aussage des Satzes vom ausgeschlossenen Dritten für endlichviele mathematische Eigenschaften bzw. für eine beliebige Spezies von mathematischen Eigenschaften [6]) als *mehrfacher Satz vom ausgeschlossenen Dritten erster* bzw. *zweiter Art* bezeichnet wird, dann ist einerseits klar, dass die Widerspruchslosigkeit des einfachen Satzes vom ausgeschlossenen Dritten keineswegs unmittelbar diejenige des mehrfachen Satzes vom ausgeschlossenen Dritten nach sich zieht, andererseits,

[4]) Uebrigens liegt bei derartigen Aeusserungen genau genommen doch wieder eine gedankenlose Anwendung des Satzes vom ausgeschlossenen Dritten, mithin eine Verdunkelung der zweiten Einsicht vor.

[5]) Vgl. F 9, S. 66 (Theorem 2).

[6]) D.h. die Existenzaussage eines Simultangesetzes, das für sie alle die Richtigkeit oder Absurdität entscheidet.

dass die im vorigen Absatze in Erinnerung gebrachte Bemerkung erst dann ihre volle Tragweite erlangt, wenn nicht nur für den einfachen Satz vom ausgeschlossenen Dritten, sondern auch für den mehrfachen Satz vom ausgeschlossenen Dritten erster Art, die Widerspruchslosigkeit feststeht. In der Tat wird im Folgenden der Beweis der letzteren Widerspruchslosigkeit erbracht. Des weiteren wird sich ergeben, dass der (für die formalistischen Hoffnungen belanglose) mehrfache Satz vom ausgeschlossenen Dritten zweiter Art keine Widerspruchslosigkeit mehr besitzt.

Die Widerspruchsfreiheit des mehrfachen Satzes vom ausgeschlossenen Dritten erster Art beweisen wir mittels vollständiger Induktion. Es sei n eine natürliche Zahl, es sei die Widerspruchsfreiheit der kombinierten Aussage des Satzes vom ausgeschlossenen Dritten für n beliebige mathematische Eigenschaften bewiesen, es seien $n + 1$ mathematische Eigenschaften $a_1, a_2, \ldots, a_{n+1}$ vorgegeben, und es sei einen Augenblick angenommen, dass die kombinierte Aussage des Satzes vom ausgeschlossenen Dritten für $a_1, a_2, \ldots, a_{n+1}$ zu einem Widerspruch führte. Das würde heissen, dass jede der 2^{n+1} Kombinationen von $a_1, a_2, \ldots, a_{n+1}$ je mit dem Richtigkeits- oder mit dem Absurditätsprädikat versehen, zu einem Widerspruch führte.

Wir behaupten, dass unter dieser Annahme der einfache Satz vom ausgeschlossenen Dritten für a_{n+1} notwendig absurd sein muss. Denn wäre a_{n+1} richtig bzw. absurd, so wäre auf Grund der Widerspruchsfreiheit der kombinierten Aussage des Satzes vom ausgeschlossenen Dritten für n beliebige mathematische Eigenschaften die Kombination der Richtigkeit bzw. Absurdität von a_{n+1} mit der kombinierten Aussage des Satzes vom ausgeschlossenen Dritten für $a_1, a_2, \ldots, a_n$ widerspruchsfrei, entgegen der Annahme des vorigen Absatzes. Die Annahme des vorigen Absatzes hat sich also als unstatthaft erwiesen, und die kombinierte Aussage des Satzes vom ausgeschlossenen Dritten für $n + 1$ beliebige mathematische Eigenschaften hat sich als widerspruchsfrei herausgestellt.

Gegenbeispiele der Widerspruchsfreiheit des mehrfachen Satzes vom ausgeschlossenen Dritten zweiter Art liefert der aus der Haupteigenschaft der finiten Mengen folgende Satz, dass bei Zerlegung des Einheitskontinuums in zwei Teilspezies eine dieser Teilspezies mit dem Einheitskontinuum identisch und die andere leer ist. Aus diesem Satze folgt nämlich, dass die kombinierte Aussage des Satzes vom ausgeschlossenen Dritten für eine beliebige Eigenschaft in bezug auf *alle* Punktkerne des Einheitskontinuums dann und nur dann widerspruchsfrei ist, wenn die Eigenschaft entweder für alle Punktkerne des Einheitskontinuums richtig oder für alle Punktkerne des Einheitskontinuums absurd ist. Insbesondere sind die beiden folgenden Aussagen kontradiktorisch :

1. *Alle Punktkerne des Einheitskontinuums sind entweder rational oder negativ-irrational.*

2. *Für alle Punktkerne des Einheitskontinuums ist die Rationalitätsfrage entweder entscheidbar oder unentscheidbar.*

Formulieren wir in Analogie mit dem Obigen folgende drei Fassungen des Prinzips der Reziprozität der Komplementärspezies:

1. *Jedes Element der Komplementärspezies der Komplementärspezies von R gehört zu R* (einfaches Prinzip der Reziprozität der Komplementärspezies).

2. *Jede endliche Spezies von Elementen der Komplementärspezies der Komplementärspezies von R gehört zu R* (mehrfaches Prinzip der Reziprozität der Komplementärspezies erster Art).

3. *Jede Spezies von Elementen der Komplementärspezies der Komplementärspezies von R gehört zu R* (mehrfaches Prinzip der Reziprozität der Komplementärspezies zweiter Art).

Alsdann besteht auch hier die Widerspruchsfreiheit nur für 1. und 2., und nicht für 3. Ein Gegenbeispiel liefert die Spezies G derjenigen Punktkerne des Einheitskontinuums C, für welche die Rationalitätsfrage entscheidbar ist. Denn die Komplementärspezies der Komplementärspezies in C von G ist mit C identisch (weil nämlich die Komplementärspezies in C von G leer ist), während wir oben gesehen haben, dass G unmöglich mit C identisch sein kann.

Mathematics. — *Beweis dass jede Menge in einer individualisierten Menge enthalten ist* [1]). By Prof. L. E. J. BROUWER.

(Communicated at the meeting of December 17, 1927).

Sei M eine beliebig vorgegebene Menge. Wir zählen zunächst die Menge der ersten Wahlen von M durch eine Fundamentalreihe ab, zählen sodann die Menge der zweiten Wahlen von M als Produkt zweier Fundamentalreihen (mittels des Diagonalverfahrens) wiederum durch eine Fundamentalreihe ab, zählen darauf die Menge der dritten Wahlen von M als Produkt der letzteren Fundamentalreihe mit einer neuen Fundamentalreihe (mittels des Diagonalverfahrens) gleichfalls durch eine Fundamentalreihe ab, usw. Alsdann bekommt jede Wahl eine neue Nummer, und wenn wir alle hierbei nach bestimmter erster, zweiter, ... bis einschliesslich n-ter Wahl für die $(n+1)$-te Wahl nicht vorkommenden Nummern als gehemmte Nummern betrachten, bekommen wir eine mit M identische „monotone" Menge N, d.h. eine mit M identische Menge N, in welcher *nach* einer ungehemmten n-ten Nummer a nur Nummern $\geqq a$ als $(n+1)$-te Nummer ungehemmt sein können, und in welcher für beliebiges n keine zwei ungehemmte n-te Wahlen mit gleichen Nummern vorkommen.

Nun nehmen wir *in der Menge* N ein Fundamentalreihe von Aenderungen $a_1, a_2, \ldots.$ vor, wobei für jedes beliebige n die Erzeugnisse der 1-sten, 2-ten, ... bis einschliesslich $(n-1)$-ten Wahl nicht von a_n beeinflusst werden. Und zwar ändern wir für a_1 zunächst in der Menge N die Reihe der ersten Wahlen derweise, dass jede gehemmte Wahl gehemmt bleibt, während eine ungehemmte Wahl dann und nur dann ungehemmt bleibt, wenn ihr in der Fundamentalreihe keine andere Wahl vorangeht, welche das gleiche Zeichen erzeugt; sodann erklären wir *nach* einer beliebigen ungehemmt *gebliebenen* ersten Wahl σ diejenigen und nur diejenigen zweiten Wahlen ungehemmt, welche zuvor nach einer das gleiche Zeichen wie σ erzeugenden ersten Wahl ungehemmt waren, und die ganze Mengenfortsetzung dieser ungehemmten zweiten Wahlen wird bei diesem „Transport" ungeändert gelassen. Hierdurch bekommen wir eine mit N identische monotone Menge N_1, in welcher gleiche Elemente immer nur aus gleichen ersten Wahlen hervorgehen.

Die Aenderung a_2 wirkt in solcher Weise auf N_1, dass für eine beliebige ungehemmte erste Wahl σ von N_1 zunächst für die auf σ folgenden zweiten

[1]) Für die Definition der Menge, der individualisierten Menge und der finiten Menge vgl. Math. Annalen **93**, S. 244—245. Im Folgenden werden wir sowohl eine beliebige Zeichenreihe wie *nichts*, kurz als „Zeichen" bezeichnen.

Wahlen die gleiche Aenderung ausgeführt wird, welche oben bei a_1 mit der Reihe der ersten Wahlen vorgenommen wurde, sodann *nach* einer beliebigen auf σ folgenden, ungehemmt *gebliebenen* zweiten Wahl τ diejenigen und nur diejenigen dritten Wahlen ungehemmt erklärt werden, welche in N_1 nach einer das gleiche Zeichen wie τ erzeugenden, auf σ folgenden zweiten Wahl ungehemmt waren, und die ganze Mengenfortsetzung dieser ungehemmten dritten Wahlen bei diesem „Transport" ungeändert gelassen wird. Hierdurch bekommen wir eine mit N und N_1 identische monotone Menge N_2, in welcher gleiche Elemente immer nur aus gleichen ersten und zweiten Wahlen hervorgehen.

Die Aenderung a_3 wirkt in solcher Weise auf N_2, dass für beliebiges σ und τ mit den auf σ und τ folgenden dritten Wahlen und ihren Mengenfortsetzungen die gleiche Aenderung ausgeführt wird, welche oben zunächst bei a_1 mit der Reihe der ersten Wahlen und ihren Mengenfortsetzungen und sodann bei a_2 mit den auf σ folgenden zweiten Wahlen und deren Mengenfortsetzungen vorgenommen wurde.

In dieser Weise bestimmen wir der Reihe nach $N_1, N_2, N_3, \ldots\ldots$, wobei jedesmal N_ν aus $N_{\nu-1}$ hervorgeht mittels Hemmung eines Teiles der vorher ungehemmten Folgen von ν Wahlen und dementsprechender Umordnung der (alle ungehemmt bleibenden) ungehemmten Folgen von $\nu+1$ Wahlen. Hierbei bemerken wir, dass einer ungehemmten Folge von $\nu+1$ Wahlen mit Indizen $\leqq m$ in $N_{\nu+1}$ eindeutig eine die gleichen Zeichen erzeugende und gleiche Indizes besitzende ungehemmte Folge von $\nu+1$ Wahlen in N_ν und der letzteren der Reihe nach in $N_{\nu-1}, N_{\nu-2}, \ldots, N_1, N$ eindeutig je eine die gleichen Zeichen erzeugende ungehemmte Folge von $\nu+1$ Wahlen mit Indizen $\leqq m$ entspricht. Mithin brauchen wir, um von Indizes $\leqq m$ besitzenden Folgen von $\nu+1$ Wahlen in $N_{\nu+1}$ die Gehemmtheit bzw. die erzeugten Zeichen festzustellen, der Reihe nach in $N, N_1, N_2, \ldots, N_\nu$ ebenfalls nur Folgen mit Indizen $\leqq m$ in Betracht zu ziehen, so dass die betreffende Feststellung einen endlichen d.h. ausführbaren Prozess darstellt.

Wenn wir nun die Menge P dadurch definieren, dass sie für jedes ν in der Wirkung der Folgen von $1, 2, 3, \ldots, \nu$ Wahlen mit N_ν übereinstimmt, dann ist die Menge P individualisiert and enthält N, also M.

Wenn wir sagen, dass zwei Mengenelemente *differieren*, wenn für passendes n die Erzeugnisse ihrer ersten n Wahlen verschieden sind, und dass zwei Mengen *übereinstimmen*, wenn keine von beiden ein von allen Elementen der anderen differierendes Element enthalten kann, so folgert man im Falle einer *finiten* Menge M mittels der Haupteigenschaft der finiten Mengen [2]) leicht, dass die zugehörige Menge P mit M übereinstimmt. *Mithin ist jede finite Menge in einer mit ihr übereinstimmenden individualisierten Menge enthalten.*

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[2]) Vgl. Math. Annalen **97**, S. 66 (Theorem 2).

Mathematics. — *A Representation of a quadrifold set of Twisted Cubics on the Points of a Linear Four-dimensional Space.* By J. W. A. VAN KOL. (Communicated by Prof. HENDRIK DE VRIES).

(Communicated at the meeting of February 25, 1928).

§ 1. The twisted cubics k^3 that pass through two given points H_1 and H_2 and cut two given lines a_1 and a_2 twice, may be represented on the points of a linear four-dimensional space R_4 in the following way. In R_4 we choose two quadratic spaces $\Omega^2{}_1$ and $\Omega^2{}_2$ that have a double line l_1 resp. l_2. We suppose a projective correspondence to be established between the points of a_1 and the planes in $\Omega^2{}_1$ and another one between the points of a_2 and the planes in $\Omega^2{}_2$. Let a curve k^3 cut a_1 in A_1 and A'_1 and a_2 in A_2 and A'_2 and let R_1, R'_1, R_2 and R'_2 be the spaces that touch $\Omega^2{}_1$ resp. $\Omega^2{}_2$ along the planes associated to the said points. To k^3 we shall associate as image point the point where the plane of intersection of R_1 and R'_1 and that of R_2 and R'_2 cut each other. Inversely an arbitrary point in R_4 is the image of one curve k^3.

§ 2. Through an arbitrary point of l_1 resp. l_2 there pass two tangent spaces of $\Omega^2{}_1$ resp. $\Omega^2{}_2$. In this way in $\Omega^2{}_1$ and $\Omega^2{}_2$ there are defined quadratic involutions of planes to which quadratic involutions of points I_1 and I_2 on a_1 resp. a_2, are associated. Each of the ∞^3 curves k^3 that cut a_1 resp. a_2 in a pair of points of I_1 resp. I_2, has its image point on l_2 resp. l_1.

l_1 and l_2 are cardinal lines; an arbitrary point P of l_1 e.g. is the image of each of the ∞^2 curves k^3 that pass through the points of a_2 which are associated to the planes where $\Omega^2{}_2$ is touched by its spaces of contact through P.

The transversal t_1 resp. t_2 of a_1 and a_2 through H_1 resp. H_2 is completed by the conics through H_2 resp. H_1 that cut a_1, a_2 and t_1 resp. t_2, to ∞^3 curves k^3 that are represented in the points of the plane of intersection σ_1 resp. σ_2 of the spaces which touch $\Omega^2{}_1$ and $\Omega^2{}_2$ in the planes associated to the points of intersection of a_1 and a_2 with t_1 resp. t_2.

There are two singular planes σ_1 and σ_2 both of which cut l_1 and l_2; an arbitrary point P of σ_1 is the image of the ∞^1 curves k^3 formed by t_1 and the conics that pass through H_2, cut t_1 and cut a_1 and a_2 in the points corresponding to the planes where $\Omega^2{}_1$ an $\Omega^2{}_2$ are touched by its spaces of contact through P which are different from the spaces of contact $l_1\sigma_1$ and $l_2\sigma_1$.

$\sigma_1\sigma_2$ is a cardinal point that represents the ∞^2 curves k^3 formed by t_1, t_2 and the transversals of a_1 and a_2.

§ 3. Our set contains ∞^2 curves k^3 that are singular for the representation, viz. the curves k^3 that cut a_1 in a pair of points of I_1 and a_2 in a pair of points of I_2. Each of these curves k^3 has ∞^1 image points, viz. all the points of a transversal of l_1 and l_2.

§ 4. $\Omega^2{}_1$ and $\Omega^2{}_2$ are the loci of the image points of the curves k^3 that touch a_1 resp. a_2.

The surface of intersection O^4 of $\Omega^2{}_1$ and $\Omega^2{}_2$ is the locus of the image points of the curves k^3 that touch a_1 as well as a_2.

§ 5. Let us investigate the representation of the system Σ_1 of the curves k^3 that have a given chord b. The curves of Σ_1 cut a_1 as well as a_2 in pairs of points of a quadratic involution. To these quadratic point involutions on a_1 and a_2 there correspond quadratic plane involutions in $\Omega^2{}_1$ resp. $\Omega^2{}_2$. These involutions have the property that two spaces which touch $\Omega^2{}_1$ resp. $\Omega^2{}_2$ in planes that correspond to each other through this involution, have a plane of intersection lying in a fixed space through l_1 resp. l_2.

The plane of intersection α_b of these spaces is apparently the image plane of Σ_1.

Two planes α_{b_1} and α_{b_2} cut each other in one point. Hence:

There is one twisted cubic that passes through two given points and has four given chords.

O^4 and α_b cut each other in four points.

There are four twisted cubics that pass through two given points, have a given chord and touch two given lines.

§ 6. Let us call the image surface of the system Σ_2 of the curves k^3 that pass through a given point P, O_P. We determine the degree of O_P by examining the intersection of it and a plane α that touches $\Omega^2{}_1$ as well as $\Omega^2{}_2$. As there is one curve k^3 of Σ_2 that passes through a given point of a_1 as well as through a given point of a_2, α cuts O_P besides in the points $\alpha\, l_1$ and $\alpha\, l_2$ in one more point. l_1 and l_2 are single lines of O_P as through two given points of l_1 and l_2 there passes one curve k^3 of Σ_2. As, accordingly, α cuts O_P in all in three points, O_P is a cubic surface. We can show that O_P has *one conic that passes through the points* $\sigma_1\, l_1$, $\sigma_1\, l_2$ and $\sigma_1\, \sigma_2$ *in common with* σ_1 *and one conic that passes through* $\sigma_2\, l_1$, $\sigma_2\, l_2$ *and* $\sigma_1\, \sigma_2$ *with* σ_2.

O_P and α_b have one point in common besides the points $\alpha_b\, l_1$ and $\alpha_b\, l_2$. Hence:

There is one twisted cubic that passes through three given points and has three given chords.

By applying the method indicated in § 8 we find that O_P and O_Q cut each other outside l_1 and l_2 in singular points only, whence:

There is no twisted cubic that passes through four given points and has two given chords.

The intersection of O^4 and O_P gives:

There are four twisted cubics that pass through three given points and touch two given lines.

§ 7. Let Ω_l be the image space of the system Σ_3 of the curves k^3 that cut a given line l. We determine the degree of Ω_l by means of the intersection with a line p that touches $\Omega_1{}^2$ as well as $\Omega_2{}^2$. p is the locus of the image points of the curves k^3 that pass through a definite point A_1 of a_1, through a definite point A_2 of a_2, and cut a_1 and a_2 outside A_1 resp. A_2 in points that correspond to each other through a certain projective correspondence between the points of a_1 and those of a_2. The number of points of intersection of p and Ω_l is, therefore, equal to twice the number of curves of Σ_3 that pass through two given points of a_1 as well as through a given point of a_2. This number is equal to two as the twisted cubics that pass through five given points and cut a given line, form a surface of the fifth degree that has triple points in the given points. Ω_l is, accordingly, of the fourth degree. We can show that l_1 and l_2 are double lines and that σ_1 and σ_2 are single planes of Ω_l.

§ 8. The intersection of Ω_l and Ω_m consists of σ_1, σ_2 and a surface O_{lm} of the degree 14, which is evidently the image surface of the system Σ_4 of the curves k^3 that cut two given lines l and m. l_1 and l_2 are quadruple lines of O_{lm} and σ_i has a curve of the sixth order that has triple points in the points $\sigma_i l_1$ and $\sigma_i l_2$ and a double point in the point $\sigma_1 \sigma_2$ in common with O_{lm}.

The intersection of O_{lm} successively with a_b and O^4 gives:

There are six twisted cubics that pass through two given points, have three given chords and cut two given lines.

There are 24 twisted cubics that pass through two given points, touch two given lines and cut two other given lines.

According to a theorem of PIERI[1]) the number of points of intersection of O_{lm} and O_P outside l_1 and l_2 is found by subtracting from the product of the degrees of O_{lm} and O_P the product of the multiplicities of l_1 on O_{lm} and O_P, the product of the multiplicities of l_2 on O_{lm} and O_P and the classes of the envelopes of the spaces through l_1 or l_2 that touch O_{lm} and O_P at the same point of one of these lines. The class of the envelope of the spaces through l_1 that touch O_{lm} and O_P at the same point of l_1, is equal to the number of spaces that pass

[1]) Rend. del Circolo Mat. di Palermo, t. V, 1891.

through an arbitrary point S and through l_1 and touch O_{lm} and O_P at the same point of l_1. It is easily proved that an arbitrary space through l_1 cuts O_P along l_1 and a conic that cuts l_1 once; accordingly this space touches O_P once, viz. in the point of intersection of l_1 and the said conic. An arbitrary space through l_1 cuts O_{lm} along the line l_1, which must be counted four times, and a curve of the tenth order that cuts l_1 in six points; consequently this space touches O_{lm} six times, viz. in the points of intersection of l_1 and the said curve of the tenth order. To an arbitrary point L_1 of l_1 we shall now associate the six points L'_1 of l_1 where O_{lm} is touched by the space that is defined by S and the plane touching O_P at L_1. Inversely through this correspondence there are associated to an arbitrary point L'_1 the four points L_1 where O_P is touched by the four spaces that are defined by S and the four planes touching O_{lm} at L'_1. The (4, 6)-correspondence between the points L_1 and L'_1 arising in this way, has 10 coincidences, hence the class in question is ten. Consequently the number of points where O_P and O_{lm} cut each other outside l_1 and l_2, is equal to $3 \times 14 - 2 . 1 . 4 - 2 . 10 = 14$. This number contains 4 points where the intersections of O_P and O_{lm} with σ_1 cut each other outside the points $l_1\sigma_1$, $l_2\sigma_1$ and $\sigma_1\sigma_2$, 4 points where the intersections of O_P and O_{lm} cut each other outside the points $l_1\sigma_2$, $l_2\sigma_2$ and $\sigma_1\sigma_2$ and the point $\sigma_1\sigma_2$ itself, which must be counted twice. There remain, accordingly, 4 points that are neither singular nor cardinal points. Thus we have found the following number, which, however, may be derived more simply in a direct way:

There are four twisted cubics that pass through three given points, have two given chords and cut two given lines.

If we apply the method indicated above to two surfaces O_{lm} and O_{no}, we find:

There are 36 twisted cubics that pass through two given points, have two given chords and cut four given lines.

§ 9. The intersection of O_{lm} and Ω_n consists of the lines l_1 and l_2, which must be counted eight times, two curves of the sixth order lying resp. in σ_1 and σ_2 and a curve k_{lmn} of the order 28 that is the image of the system Σ_5 of the curves k^3 that cut three given lines l, m and n. k_{lmn} cuts l_1 and l_2 in 14 points, as the number of points of intersection of k_{lmn} and l_1 as well as the number of points of intersection outside l_1 of k_{lmn} and a tangent space of Ω^2_1 is equal to the number of curves of Σ_5 that pass through a given point of a_1. The number of points of intersection of k_{lmn} and σ_1 is equal to the number of conics that pass through H_2 and cut the six lines a_1, a_2, t_1, l, m and n (in different points). The conics that pass through H_2 and cut a_1, a_2, l, m and n form a surface of the degree 18 [1]) that is cut by t_1 outside the points of inter-

[1]) Cf. SCHUBERT, Kalkül der abzählenden Geometrie, p. 96, where the numbers of conics $P\nu^6 = 18$ and $P^2\nu^4 = 4$ are derived.

section of t_1 with a_1 and a_2, which are quadruple lines of the surface, in ten points. Accordingly σ_1 and σ_2 are cut by k_{lmn} in ten points.

The intersection of $\Omega^2{}_1$ and k_{lmn} gives:

There are 28 twisted cubics that pass through two given points, have a given chord, cut three given lines and touch another given line.

§ 10. We can further investigate the representations of several other systems, as the systems of the curves k^3 that touch one, two or three given planes, that touch a given plane and at the same time cut one or two given lines, that touch a given plane and at the same time pass through a given point and others.

The numbers that may be deduced in this way and those already found above are the following ones:

$P^4B^2 = 0$	$P^3B^2\nu^2 = 4$	$P^2B^3\nu^2 = 6$	$P^2B^2\nu^4 = 36$
$P^3B^3 = 1$	$P^3B^2\nu\varrho = 8$	$P^2B^3\nu\varrho = 12$	$P^2B^2\nu^3\varrho = 72$
$P^2B^4 = 1$	$P^3B^2\varrho^2 = 16$	$P^2B^3\varrho^2 = 24$	$P^2B^2\nu^2\varrho^2 = 144$
$P^3T^2 = 4$			$P^2B^2\nu\varrho^3 = 288$
$P^2BT^2 = 4$			$P^2B^2\varrho^4 = 576$
$P^3BT\nu = 4$	$P^2B^2T\nu = 4$	$P^2T^2\nu^2 = 24$	$P^2BT\nu^3 = 28$
$P^3BT = 8$	$P^2B^2T\varrho = 8$	$P^2T^2\nu\varrho = 48$	$P^2BT\nu^2\varrho = 56$
		$P^2T^2\varrho^2 = 96$	$P^2BT\nu\varrho^2 = 112$
			$P^2BT\varrho^3 = 224$

Here P indicates the condition that a twisted cubic pass through a given point, B that it have a given chord, ν that it cut a given line, T that it touch a given line and ϱ that it touch a given plane.

§ 11. From the above we can derive properties of different surfaces formed by systems of ∞^1 curves k^3 [1]).

The curves k^3 that touch a_1 and a_2 and cut a given line l, form a surface of the degree 24 that has 12-fold points in H_1 and H_2; a_1 and a_2 are eightfold lines and l is a quadruple line of this surface.

The curves k^3 that touch a_1 and cut two given lines l and m, form a surface of the degree 28 that has 14-fold points in H_1 and H_2; a_1 is an eightfold line, a_2 is a twelvefold line and l and m are quadruple lines of this surface. Etc.

[1]) Cf. also these Proceedings **30**, p. 1016 (1927).

Bacteriology. — *On the Transmuted Tubercle bacilli type BTTx., and their significance for the Diagnosis and the Therapy of Tuberculosis* By C. H. H. SPRONCK and W. HAMBURGER.

(Communicated at the meeting of October 29, 1927).

The classical law of the invariableness of bacteria had been advanced under the influence of the great discoveries of ROBERT KOCH. But gradually facts became known that negatived this theory. Morphological and biological properties may be lost, and new ones may be acquired. At present the variability of bacteria is wellnigh an ascertained fact; it is considerable with some species, small with others. But bacteria are by no means kaleidoscopical beings, their invariability being subject to laws still unknown.

According to NEUFELD [1]) the tubercle bacillus belongs to the non-fluctuating bacteria. But it has long been known, that the tubercle bacillus possesses a great adaptive power, and that it is possible to cultivate it on all sorts of media. Now, we detected that on media that are gradually altered and become poorer and poorer, the properties of the parasite alter considerably at a given moment. When we saw this for the first time, we thought of course of contamination, and months passed by before we got the conviction, that we had to do with a variety of the parasite. Small though the variability of the tubercle bacillus may seem, in reality also this micro-organism is undoubtedly capable of change, and of an intense change.

The stability of the transmuted parasite seems even to be still greater than that of the typical tubercle bacilli. Anyhow, in spite of all our efforts we have not yet succeeded in inducing the transmuted parasite to resume the state of a typical tubercle bacillus or of another variety. Moreover, the transmuted parasite has not only lost certain properties, it has also acquired new ones, which is the reason why we have not spoken of modified but of transmuted tubercle bacilli, and have styled the transmuted tubercle bacillus bacillus tuberculosis transmutatus x (BTTx) [2]).

Compared with typically human and bovine strains the transmutant grows with amazing rapidity, but only at 37—38° C. and in the presence of oxygen. It does not grow at room-temperature.

Already in bacilli, only 24—48 hours old that do not show granulation,

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[1]) Deutsche med. Wochenschr.: 1924, N⁰. 1.

[2]) JOLLOS (Zentralbl. f. Bakteriol. Abt. I, Orig. Bd. **93**), indeed, has cogently maintained that in the case of bacteria it is not right to speak of mutation, because with asexual propagation it is hardly possible to judge of heredity, and as the bacteria do not possess a nucleus, the cytological analysis is out of the question. We might use LEHMAN's term "Klonumwandlung", but most bacteriologists use the word mutation, which is understood in all countries.

an oval, protruding spore may appear, which always lies near one of the extremities of the rod. In one and the same cell we never find more than one spore. Before long the spore is mature, the rod degenerates and the spore gets free. The elements of the transmutant, (threads, bacilli, and spores) are non-acidfast, neither do they stain after GRAM. Only the younger still immature spores stain occasionally after GRAM. The spores do not possess great resistance, still they seem to be instrumental in preventing the destruction of the culture by heating to 56—59° C. during one hour, even when before the heating 0.5 % carbolic acid has been added. The transmuted bacilli, therefore are at least as resistant as the typical bacilli. If the latter are killed in a solution of 0.01 % formol within 24 hours (CALMETTE) the transmuted bacilli are undoubtedly much more resistant. Cultures of typical tubercle bacilli protected from exsiccation and light, survive at the most for 1½ years when kept in a cool place ; but most cultures die off much sooner. Under the said conditions cultures of the transmuted tubercle bacillus appeared without exception capable of reproduction after 1½ years.

The transmutant has completely lost virulence for caviae. It does not produce tuberculin, nor is the substance contained in its elements. Whereas typical tubercle-bacilli are rich in fat and waxy substances, the transmuted ones are very poor in this respect.

Thus far we have transmuted 7 strains of tubercle bacilli, viz. 6 human ones and 1 bovine. The transmutation never having failed, no matter whether the strain had been cultivated in the laboratory already for some time, or was completely fresh, we consider the transmutation of every strain possible, and that of the third type of virulent tubercle bacilli, viz. bacillus gallinaceus we deem probable. For the transmutation, which in our laboratory is sometimes called emaciation-curve, at least 2 or 3 months are required.

It is remarkable that always the same transmutant came forth from 7 strains. True, there are differences among the transmuted strains inter se, but they are of a quantitative character. One strain grows quicker than the other, also the agglutinability varies.

It has long been known, that in cultures of typical tubercle bacilli not only acid-fast but also non-acid-fast rods are observed. They were suspected to be degenerated or imperfect rods of KOCH, or young, still undeveloped bacilli and they were even considered as a variety. In BALDWIN, PETROFF and GARDNER's [1]) recently published work "Tuberculosis" we read : "All attempts to separate non-acid-fast from acid-fast forms have failed, so that they must be considered to represent immature or imperfect forms rather than a separate variety".

We consider the non-acid-fast rods of the transmutant to be similar to those of the cultures of the typical tubercle-bacillus, but we assume the latter to be part of the parasite, which is not a simple

[1]) The Trudeau Foundation Studies, 1927, p. 33.

schizomycete, but a higher micro-organism, which belongs to the genus actinomyces. In its growth outside the infected organism it forms two sorts of rods, acid-fast and non-acid-fast, both originating from non-acid-fast threads and multiplying through fission. On the best media the parasite forms few non-acid-fast rods, but their number increases when the medium grows poorer, and now at one moment mutation takes place which under the modified circumstances largely benefits the existence and the propagation. It reminds us of a self-defence in distress.

We have decidedly observed the same mutation, but till now only once, after inoculating tissue of a tuberculous cavia-spleen on a poor medium, absolutely unfit to grow tubercle bacilli. After the medium, which had been guarded against exsiccation, had remained at 38° C. for two months without evincing any sign of growth, a suspicious speck became visible that gradually got bigger. Of course we suspected contamination, the more so since the culture exhibited a number of red specks, which had never been observed in the transmutant. On closer inspection we learned that it really was the transmutation, which did not exhibit the red spots any more in the next following culture.

Long before the appearance of the transmutant a typical culture had been obtained from the same spleen tissue, and the transmutation was commenced in the ordinary way. The transmutant thus obtained, and the one cultivated directly from the tuberculous spleen-tissue, could initially be distinguished by the latter's slower growth and smaller agglutinability. However, these differences stayed away in further cultivation.

In the tuberculous spleen-tissue that had been spread on the surface of the poor medium processes had presumably been going on that were analogous to those on the gradually altered media that are used for the transmutation. We assume that in the spleen-tissue a small typical culture has arisen that was soon short of foodstuffs ; that the parasite adjusting itself by little and little to this shortage, had passed into the transmutant, which thrives better on the poor medium than the typical parasite on the best media hitherto known. The direct, or rather quasi direct culture of the transmutant is now tried again and again, but as yet without success.

Of late years the properties of the transmuted tubercle bacillus have been examined in different directions. It appeared thereby that they are significant for the diagnosis as well as for the therapy of tuberculosis.

After our communication concerning the transmutant as a tuberculosis-diagnosticum, the agglutination-titre of the blood-serum has been examined of another 100 subjects, suffering from, or suspected to suffer from tuberculosis, of patients suffering from other diseases, or apparently healthy persons. The titre was determined as in the first investigation. Only in some cases has the titre been established more precisely.

The cases of indubitable tuberculosis have now been divided into the following groups : 1°. sufferers from tuberculosis of the skin and the mucous membranes of mouth, throat, and nose (Table I), 2°. sufferers

from tuberculosis of lymphatic glands, bones, and joints (Table II), 3⁰. sufferers from pulmonary tuberculosis (Table III) and 4⁰. sufferers from abdominal-tuberculosis. In Table V the cases are reported, in which we suspected the existence of tuberculosis ; in Table VI sufferers from other diseases or apparently healthy people. In all the tables the cases have again been arranged according to the titre the better to determine the relation between the number of positive titres (1 : 100 and higher) and that of the negative ones.

The new observations justify us as yét in considering 1 : 100 as the lowest titre-limit of the positive reaction.

TABLE I. Tuberculosis of skin and mucous membranes.

N⁰.	Diagnosis	Titre	Notes
1	Scrophuloderma colli. (Prof. v. D. VALK, Groningen.)	1 : 10	About 2 years ago begun with a swollen gland. Patient looks healthy and now leaves clinic quite cured.
2	Tuberculosis cutis verrucosa. (Prof. v. D. VALK, Groningen.)	1 : 10	Verrucae at left middlefinger and fore-arm.
3	Lupus vulgaris nasi. (Prof. v. D. VALK, Groningen.)	1 : 50	Boy of 12 years, pale, small, slender; no fever. For three years lupus of nose and cheek. Röntgenization: large calcified hilus-glands and a calcified focus in the right inferior lobe.
4	Lupus faciei, mucosae oris et nasi. (Prof. v. D. VALK, Groningen.)	1 : 50	Boy 17 years old, lupus for three years. Pirquet very strong +.
5	Lupus nasi. (Prof. BENJAMINS, Groningen.)	1 : 50	Woman 38 years old.
6	Lupus vulgaris faciei. (Prof. v. D. VALK, Groningen.)	1 : 100	
7	Lupus vulgaris disseminatus. (Prof. v. D. VALK, Groningen.)	1 : 100	Face and arms.
8	Lupus vulgaris in scrophuloderma-scars. (Prof. v. D. VALK, Groningen.)	1 : 100	
9	Lupus. (Prof. BENJAMINS, Groningen.)	1 : 100	Boy 15 years old.
10	Lupus faciei et mucosae nasi. (Dr. LA CHAPELLE, Assen.)	1 : 100	Vocal chords also affected. Lupus of 6 years standing, still small foci of it are persisting.
11	Lupus. (Prof. BENJAMINS, Groningen.)	1 : 200	Girl of 16 years.
12	Lupus mucosae nasi. (Prof. v. D. VALK, Groningen.)	1 : 200	v. Pirquet +.
13	Lupus. (Prof. BENJAMINS, Groningen.)	1 : 300	Woman 49 years old.

TABLE II. Tuberculosis of lymphatic glands, bones, joints.

N⁰.	Diagnosis	Titre	Notes
1	Tuberculosis of the knee-joint. (Dr. LA CHAPELLE, Assen.)	1 : 10	Röntgenization; focus in the epiphysis of the femur. The aspirated fluid gave a positive cavia test.
2	Tuberculous jugular glands. (J. HOOGKAMER, The Hague.)	1 : 10	
3	Tuberculosis of the ileo sacral joint. Formation of abscess. (Dr. LA CHAPELLE, Assen.)	1 : 50	25 years, general condition excellent. Formerly glandular abscess at the neck.
4	Tuberculosis of the left foot, perforation of the outer ankle. (Dr. LA CHAPELLE, Assen.)	1 : 100	21 years; formerly pleuritis.
5	Tuberculosis of jugular glands.	1 : 100	22 years, also laryngitis tuberculosa sputum: cavia-test not yet ended.
6	Tub. of right elbow and malum Pottii. (M. C. A. KLINKENBERGH and TH. VAN DEN EIJDEN, Utrecht.)	1 : 100	Girl of 16 years. Processes are coming to rest.
7	Coxitis tub. many fistulae. (M. C. A. KLINKENBERGH and TH. VAN DEN EIJDEN, Utrecht.)	1 : 100	Process seems to be active.
8	Tuberculosis jugular glands with abscesses. (Dr. LA CHAPELLE, Assen.)	1 : 100	Besides old tub. of the lungs.
9	Tub. of metacarpus of the left hand; fistulae. (M. C. A. KLINKENBERGH and TH. VAN DEN EIJDEN, Utrecht.)	1 : 100	Girl of 14 years. No fever.
10	Tub. of the tarso metatarsal joint I; fistulae. (M. C. A. KLINKENBERGH and TH. VAN DEN EIJDEN, Utrecht.)	1 : 100	Girl of 22 years. Resection 8 Aug. '27. Remains small fistula and pressure-pain. Titre determined 29 Sept. '27.
11	Tub. of the left large trochanter. (M. C. A. KLINKENBERGH and A. VAN BEEK, Utrecht.)	1 : 100	Man 49 years has been recently operated.
12	Most probably tuberculous coxitis. (Dr. LA CHAPELLE, Assen.)	1 : 200	Boy of 5 years, photo atrophy, no focus, v. Pirquet first time —, second time strong +.
13	Coxitis tuberculosa, cured. (Dr. LA CHAPELLE, Assen.)	1 : 200	26 years, photo only morbid growth of bone.
14	Spondylitis tuberculosa. (J. HOOGKAMER, The Hague.)	1 : 200	29 years. Formerly symptoms of tub. of the lungs.
15	Spondylitis tuberculosa abscess descended along carious ribs. (Dr. LA CHAPELLE, Assen.)	1 : 200	29 years. Formerly tub. of the lungs.
16	Osteomyelitis tuberculosa. (J. HOOGKAMER, The Hague.)	1 : 200	Woman 40 years, also tub. of the lungs, lingering for years.
17	Gonitis tuberculosa sin. (M. C. A. KLINKENBERGH and A. VAN BEEK, Utrecht).	1 : 200	Man 33 years, is feverish.
18	Tub. of the hand. (Dr. LA CHAPELLE, Assen.)	1 : 300	37 years, recovering v. Pirquet very weak positive.
19	Tuberculous jugular glands, fistulae. (Dr. LA CHAPELLE, Assen.)	1 : 300	40 years; fistulae from his 19th year. For the rest healthy.

With a titre of 1 : 100, we may think of the resorption of millions of tubercle bacilli, so that there is no question about an insignificant tuberculous focus, but of a progressive tuberculous process, that exists or has existed for some time, for the production of antibodies proceeds still some time after the resorption of the antigens has stopped. On the other hand the titre 1 : 100 does not indicate anything about the nature and the extent of the tuberculous process. Not infrequently a small focus contains far more tubercle bacilli than a large one and the resorption is larger at one place than at the other.

From a practical point of view it is significant that already in incipient

TABLE III. Tuberculosis of the Lungs.

Nº.	Diagnosis	Titre	Notes
1	Chronic pulmonary tub. (W. v. HASELEN, IJmuiden and S. v. SLOOTEN, Haarlem).	1 : 10	Severe, hopeless case.
2	Tub. of the Lungs. (L. WEIJL, Middelburg).	1 : 10	Far advanced.
3	Chronic extensive tub. of the lungs. (M. LE HEUX, Doorn).	1 : 50	Boy of 18 years. Far advanced.
4	Chronic tub. of lungs and larynx. (J. HOOGKAMER, Den Haag.)	1 : 100	26 years, far advanced, pleuritis and spina ventosa.
5	Chronic tub. of the lungs. (DR. KERSBERGEN and C. J. BOUWER, Haarlem).	1 : 100	
6	Chronic tub. of the lungs. (DR. KERSBERGEN and C. J. BOUWER, Haarlem).	1 : 100	
7	Chronic tub. of the lungs. (DR. KERSBERGEN and C. J. BOUWER, Haarlem).	1 : 100	
8	Chronic tub. of the lungs. (J. BURCK, 's Graveland).	1 : 100	29 years.
9	Chronic tub. of the lungs. (F. A. v. D. BREGGEN, Haarlem).	1 : 100	
10	Chronic tub. of the lungs. (J. HOOGKAMER, The Hague.)	1 : 100	Tubercle bacilli in sputum.
11	Chronic tub. of the lungs. (J. WACHTERS, Culemborg).	1 : 100	Also coxitis tuberculosa and abdominal tub.
12	Tub. of the lungs (formerly). (DR. STENVERS, Utrecht).	1 : 100	Now influenza and acute encephalitis.
13	Chronic tub. of the lungs. (Dr. LA CHAPELLE, Assen.)	1 : 200	Besides tub. cutis at neck and left shoulderblade. Now hydrops genu after trauma; titre of the aspirated fluid 1 : 50.

tuberculosis of the lungs the titre is 1 : 100 or higher. Because recent lung-foci generally contain many tubercle bacilli and there is in the lungs a favourable opportunity for resorption, it is easily understood that the titre is high already when the symptoms are still vague.

In rapidly progressing exsudative forms of pulmonary tuberculosis the titre has not yet been examined. Probably it will appear to be low, because here the blood is easily overcharged with antigens, which the organism cannot destroy. Table III again shows, that in lingering forms the titre is regularly high and remains so for a long time; it falls only when the disease is far advanced, so that a serological diagnosis is superfluous.

Also when the lung-process has come to rest or is cured, a fall of the titre is sure to reveal itself after some time. That in the cases mentioned the titre is lower than 1 : 100 does not lessen the practical significance of the agglutination-test for the recognition of the pulmonary tuberculosis If we compare the low titre with the clinical observation, its significance can easily be interpreted.

Probably a titre of 1 : 100 is also attained in a short time in the case of tuberculosis of the peritoneum (Table IV). The large surface of the peritoneum and the initially unimpeded resorption are favourable factors.

TABLE IV. Tuberculosis of the peritoneum.

N⁰.	Diagnosis	Titre	Notes
1	Peritonitis tuberculosa. (Dr. LA CHAPELLE, Assen.)	1 : 50	Much fluid in the belly. Also both lungs diffusely affected. High temperatures, v. Pirquet —.
2	Tubercles on the bowels. (Dr. LA CHAPELLE, Assen.)	1 : 90	Found at an operation of the abdomen, probably originated a short time ago in the lactation-period.
3	Tub. peritonei. (Dr. LA CHAPELLE, Assen.)	1 : 100	Previously pleuritis, fever, v. Pirquet +.
4	Tub. of the abdomen. (Dr. LA CHAPELLE, Assen.)	1 : 200	Recovering, v. Pirquet weak +.
5	Peritonitis t.b.c. (DR. LA CHAPELLE, Assen.)	1 : 300	Regressive process.

During an operation of the abdomen Dr. LA CHAPELLE detected tubercles in the serosa of the ileum, that had probably originated in the lactation-period and had not yet induced any sympton. Nonetheless the titre rose as high as 1 : 90 (Table IV n⁰ 2). The severe far advanced case n⁰. 1 (tuberculosis of abdomen and lungs) shows a fall of the titre (1 : 50) accompanied by negative tuberculin-reaction (negative anergy of v. HAYEK).

With lupus the tubercle bacilli are extremely rare. In the skin temperature and light are detrimental to their reproduction. It cannot be expected, there-

fore, that a small lupus-spot, a dissection-tubercle, a tuberculosis verrucosa cutis would induce a titre of 1 : 100. In 13 cases of lupus and other forms of tuberculosis cutis (Table I) the titre was lower than 1 : 100 in four cases (30 %). In the other cases it varied from 1 : 100 to 1 : 300.

Also in tuberculosis of lymphatic glands, bones and joints (Table II) the titre can be lower than 1 : 100. The resorption is decidedly tardier here, and the number of tubercle bacilli is as a rule smaller than in the lungs, so that the whole organism cannot join so soon in the strife against the local process. In 3 (15 %) of the cases examined the titre was lower than 1 : 100, in the others it varied from 1 : 100 to 1 : 300.

In case N⁰. 1, a patient of Dr. LA CHAPELLE (Assen) suffering from a tuberculous inflammation of the knee-joint, the titre appeared to be smaller than 1 : 10 on the first examination, and more than a month later it had risen only little (rather more than 1 : 10). Röntgenization distinctly revealed a tuberculous focus in the epiphysis of the femur and by puncture he obtained a little viscous fluid, in which tubercle bacilli could be demonstrated with the aid of the cavia-test. But the liquid injected subcutaneously on August the 10th contained only few bacilli, as in August and in September no symptom of tuberculous infection was noticeable. Not before October did a small infiltrate appear on the place of the infection and some time later swelling of the regionary glands occurred.

From Table V it appears again that in cases that remind us of tuberculosis the agglutination-test often turns out positive, which speaks for its validity. Out of the 36 sera of this group 21 (58 %) had a titre of 1 : 100 to 1 : 300. It goes without saying that we tried to verify the serological diagnosis. In one case there were afterwards distinct, clinical manifestations of tuberculosis (n⁰. 17). In another (n⁰. 22) tubercle bacilli were found in the urine, and the extirpated kidney appeared to be tuberculous. In a third (n⁰. 32) the cavia-test demonstrated tubercle bacilli in the sputum.

Altogether we examined up to now $36 + 33 = 69$ doubtful cases with $21 + 17 = 38$ (55 %) positive results.

The group of 20 sufferers from divers other diseases or apparently healthy persons. (Table VI) gives in as many as 7 cases (35 %) a positive result. It is as if this high percentage renders the practical significance of the agglutination-test as illusive as that of the positive tuberculin-reaction. On closer inspection of the positive cases, however, the surmise rises that the experiment might contribute to the recognition of the tuberculous character of diseases, whose cause is still unknown, or the predisposing influence which a tuberculous focus exerts on the origin of other diseases.

Since much has been written about the connection between erythema induratum of BAZIN and tuberculosis and JADASSOHN considers this disease even as a form of tuberculosis cutis, we looked for cases of this disease and thanks to the assistance of Dr. TER MATEN we had an opportunity to examine the blood-serum in two cases. In the one case (n⁰. 16) the titre was 1 : 200, in the other 1 : 50 (n⁰. 12). Truth to tell, we had expected a

TABLE V. Tuberculosis?

N⁰.	Diagnosis	Titre	Notes
1	Tuberculosis? (J. HOOGKAMER, The Hague.)	1 : 10	After an operation irregular fever.
2	Vague complaints. (DR. ENKLAAR, Amsterdam).	1 : 10	
3	Throat complaints, headache struma. (J. HOOGKAMER, The Hague.)	1 : 10	Tuberculous relations.
4	Divers complaints, not specific laryngitis. (J. HOOGKAMER, The Hague.)	1 : 10	
5	Suspicious cutaneous inflammation. (J. HOOGKAMER, The Hague.)	1 : 10	
6	Presumably no tub. (J. HOOGKAMER, The Hague.)	1 : 10	
7	Suppuration of the ear after radical operation. Tuberculosis? (J. HOOGKAMER, The Hague.)	1 : 10	Radical operation years ago. No reason to think of tuberculosis.
8	Headache. (W. HAMBURGER, Utrecht).	1 : 50	
9	Chronic rheumatic pedarthritis of Poncet. (Dr. LA CHAPELLE, Assen.)	1 : 50	X-photo does not give a decision.
10	Tuberculosis(?). (Dr. LA CHAPELLE, Assen.)	1 : 50	Boy of 6 years. Subcutaneous, granulating cavity in the middle under the chin.
11	Process in the thoracic cavity. Tuberculosis? (Dr. LA CHAPELLE, Assen.)	1 : 50	Patient has been attended in a sanatorium, has had a pleurisy, that was soon cured. Now increasing oppressiveness, cause unknown.
12	Knee irritated through paving, meniscus-luxatio. (Dr. LA CHAPELLE, Assen.)	1 : 50	Röntgenization negative.
13	Tuberculosis of the lungs? (J. WACHTERS, Culemborg).	1 : 50	
14	Tuberculosis? (Dr. LA CHAPELLE, Assen.)	1 : 50	Woman 29 years, abdominal complaints. Scars in the neck of glands in youth.
15	Tuberculosis? (J. HOOGKAMER, The Hague.)	1 : 70	The one physician thinks of tub. the other does not.
16	Tuberculosis? (DR. KERSBERGEN and C. J. BOUWER, Haarlem).	1 : 100	
17	Tuberculosis? (DR. ENKLAAR, Amsterdam).	1 : 100	Serological diagnosis, afterwards confirmed.
18	Presumably tub. of the lungs. (J. HOOGKAMER, The Hague.)	1 : 100	
19	Tabes mesaraica. (DR. E. H. B. VAN LIER, Utrecht).	1 : 200	20 years old.

TABLE V. Tuberculosis? *(continued).*

N⁰.	Diagnosis	Titre	Notes
20	Rise of temperature, Cause unknown. (J. HOOGKAMER, The Hague.)	1 : 100	
21	Bronchial asthma. Tuberculosis? (J. HOOGKAMER, The Hague.)	1 : 100	
22	Tub. of the kidney? (J. WESTENBURG, Bloemendaal).	1 : 100	Urine appeared to contain tubercle bacilli (cavia-test). The extirpated kidney was found to be tuberculous.
23	Rheumatic pains, reminding of Poncet. (Dr. LA CHAPELLE, Assen.)	1 : 100	
24	Tuberculous suppuration of the ear? (J. HOOGKAMER, The Hague.)	1 : 100	
25	General weakness.	1 : 100	Girl 18 years. Feels ever tired and weak in the muscles.
26	Chronic bronchitis. Tuberculosis. (DR. FREERICKS, 's Hertogenbosch).	1 : 100	No tubercle bacilli in the sputum (cavia-test).
27	Incipient tub. of the lung(?). (CHR. EGGINK, Amersfoort).	1 : 100	
28	Tuberculosis? (Dr. LA CHAPELLE, Assen.)	1 : 100	Girl, 11 years, weak, colipyelitis. v. Pirquet strong +.
29	Incipient tub. of the lungs. (J. WACHTERS, Culemborg).	1 : 100	
30	Painful knee, slight swelling of the capsule. (M. C. A. KLINKENBERGH and TH. VAN DEN EIJDEN, Utrecht.)	1 : 100	Girl of 15 years.
31	Tubercle under the retina? Hoarseness, inflammation of epiglottis and larynx-tuberculosis? (J. HOOGKAMER, The Hague.)	1 : 100	
32	Asthma bronchiale. Tuberculosis? (J. HOOGKAMER, The Hague.)	1 : 200	The cavia-test confirms the serol. diagn.
33	Incipient tub. of the lung? (Our personal observation).	1 : 200	Young woman, debility, fatigue, small rise of temperature. Does not track or cough. Lungs carefully examined by internists. Doubt about affection of the apex. Röntgenization negative.
34	Suspicious pulmonary disease. (J. BAST, Oudehaske).	1 : 200	First cavia-test (sputum) negative; second one not ended yet.
35	Otitis media perforativa suppurativa for years. (J. HOOGKAMER, The Hague.)	1 : 200	This patient is the mother of No. 25. Previously she was glandulous.
36	Bilateral suppuration of the ear. Tuberculosis? (J. HOOGKAMER, The Hague.)	1 : 300	Previously patient was glandulous.

TABLE VI. Suffers from other diseases, or apparently healthing.

N⁰.	Diagnosis	Titre	Notes
1	Lupus erythematosus. (E. SWAAB, Breda).	1 : 10	Diagnosis made by several dermatologists.
2	Sepsis. (L. ALKEMADE, St. Oedenrode).	1 : 10	Purulent infiltration of the lower leg appeared to be rich in micrococcus catarrhalis.
3	Psoriasis. (J. HOOGKAMER, The Hague.)	1 : 10	No trace of tuberculosis.
4	Apparently healthy. (Our personal observation).	1 : 10	Robust youth.
5	Chronic intestinal phenomena. (W. HAMBURGER, Utrecht).	1 : 10	Nervous, weak man.
6	Sterile abscess between hepar and pancreas. High temperature. (Dr. LA CHAPELLE, Assen.)	1 : 10	Afterwards pleurisy, clear sterile liquid.
7	Hoarseness. (J. HOOGKAMER, The Hague.)	1 : 10	Vocal chords do not close up well. (pression on nervus recurrens?) No symptoms of larynx- or lung-tuberculosis.
8	Morbus Basedowi. (J. HOOGKAMER, The Hague.)	1 : 10	
9	Ozena. (J. HOOGKAMER, The Hague.)	1 : 50	
10	Lues. (J. HOOGKAMER, The Hague.)	1 : 50	
11	Throat complaints. (J. HOOGKAMER, The Hague.)	1 : 50	
12	Erythema induratum Bazin. (Dr. TER MATEN, Amsterdam.)	1 : 50	
13	Chronic Bronchitis. (J. HOOGKAMER, The Hague.)	1 : 50	74 years old.
14	Spondylitis type Bechterew; light psoriasis. (Dr. LA CHAPELLE, Assen.)	1 : 100	
15	Psoriasis and rheumatic complaints. (W. HAMBURGER, Utrecht.)	1 : 100	51 years; no tub. in the family.
16	Erythema induratum Bazin. (Dr. TER MATEN, Amsterdam.)	1 : 200	The father of the 32-year-old patient died of haemoptoe.
17	Ozena. (P. BUTTER, Kampen.)	1 : 200	Girl, 22 years. For the rest quite healthy, the 24-years-old sister (also ozena) has been suffering from a light tub. of the lung. Parents and 7 brothers and sisters healthy.
18	Apparently healthy. (J. HOOGKAMER, The Hague.)	1 : 200	Miss B. no reason to think of tub. Tub. not in the family.
19	Apparently healthy. (J. HOOGKAMER, The Hague.)	1 : 200	Afterwards a kidney-process was detected. Examination of the urine: cavia-test 5 Sept. up to now no manifestations of tub. The kidney already extirpated had large purulent cavities. Result of anatomic examination not yet known.
20	Rheumatism. (J. HOOGKAMER, The Hague.)	1 : 300	Suffering for a long time. Treated with all sorts of remedies without a positive result, also with strepto-staphylococci vaccin.

positive result in both cases. The high titre in case N⁰. 16 cannot be at all surprising.

Concerning the 22-year-old ozena-sufferer N⁰. 17, titre 1 : 200, for the rest in rude health, we heard that her sister, older by two years, who also suffers from ozena, is less healthy and has passed through a light tuberculosis of the lungs, so that the high titre is probably the consequence of a favourably regressing or an arrested tuberculous process

As for case N⁰. 19, titre 1 : 200, a diseased kidney was detected afterwards. The examination of the urine is not yet terminated (cavia-test), but in the mean time the kidney has been removed. It showed large suppurating hollows. The result of the anatomic investigation is not yet known.

There remain still 4 cases (20 %) whose high titres cannot be accounted for as yet. But it is remarkable that of these persons one is suffering from chronic rheumatism, one from rheumatism and psoriasis, and one from BECHTEREW's type of spondylosis and light psoriasis.

Also in our first investigation in the group of 35 subjects suffering from divers diseases or apparently healthy people, there were three inexplicable cases two of which were rheumatism.

Thus far 9 cases of rheumatism have been examined, in six of them a titre was found of 1 : 100 or higher (in one case a titre of 1 : 10 and in two cases of 1 : 50). We may think that rheumatism and tuberculosis may occur side by side, or perhaps of tuberculous rheumatism of PONCET. We may also think of para-agglutinins.

In our previous agglutination-tests with acidfast tubercle bacilli para-agglutinins were very cumbersome. When using the transmutant nothing was noticed as yet of the very high titres, previously revealed in other diseases and also in tuberculosis. In tuberculosis we, therefore, expected some times titres of 1 : 500 and higher; a higher titre than 1 : 400 (one case) has however not been found as yet.

So far as we know the literature does not contain any indication that formerly difficulties have been met with in chronic rheumatism, so that it seems expedient to determine the titre for a larger number of cases. It is not out of the bounds of probabilities, that the transmutant points to tuberculous cases in the chaos of chronic rheumatism. Should para-agglutinins appear to come into play, this will detract little from the value of the diagnostic agglutination-test, since pulmonary tuberculosis and chronic rheumatism seldom go together.

In virtue of our personal investigation the rather weak and transient tuberculosis-immunity can be analyzed into two components: an allergic (hyper-sensitiveness to tuberculin) and an antitoxic-anti-infectious one, which induce, the one as well as the other, a small degree of immunity in the cavia. This immunity is so weak, that, if the infectious dosis is only a little larger than the dosis minima, little or nothing can be noticed of a greater power of resistance in the previously treated animals.

In the transmuted tubercle bacilli the antigen of the tuberculin-allergy is altogether absent. They evoke in the cavia only a bacillar protein-allergy, which probably has nothing to do with the tuberculosis-immunity. On the other hand they do contain antigens of the antitoxic-anti-infectious component.

To our mind there is often in the tuberculous patient a shortage of either component, while most colleagues will be inclined to assume only a shortage of the antitoxic-anti-infectious component. The reason of these different views is that some consider great sensitiveness to tuberculin others the absence of this allergy as a favourable sign. We believe that the allergy is evoked by an antibody, which is eagerly absorbed by the cells, so that there is often a paucity of this substance in the blood. So the tuberculin-reaction occurs in cells, and the positive reaction is in our opinion a sign that the blood contains little antibody. If on the contrary the blood is rich in antibody, the tuberculin-adsorption takes place not within but outside the cells, and we speak of negative reaction. So, in our opinion, the negative reaction is preferable to the positive, except in case of negative anergy (v. HAYEK).

A vaccine, consisting of killed typical tubercle-bacilli would be preferable, because all the antigens are contained in it. But vaccinetherapy cannot employ typical tubercle bacilli, because the blood is most often poor in the antibody of the allergic component, so that dangerous adsorption processes are to be feared in the tuberculous cells that are richest in this antibody, and the subcutaneous injection also evokes infiltrations and abscesses due to the very difficult resorption of bacilli rich in fat.

To our mind, therefore, the use of transmuted tubercle-bacilli is making a virtue of necessity, for they contain only antigens of the antitoxic-anti-infectious component. The same drawback is observed in the tuberculin-therapy, which applies only the antigen or rather the haptene of the allergic component, and consequently can reinforce only the allergic component.

A vaccine consisting of transmuted tubercle bacilli and tuberculin would, therefore, be preferable, but we thought it expedient in the first place to apply the transmuted tubercle bacilli as such, in order to ascertain whether they are, indeed, active, and if so, to what extent.

We used exclusively bacilli 2×24 hours old that had been killed with formol (0.2 %). Very likely the use of living bacilli would be better, but seeing that the resistance is not inconsiderable and the parasite of the tuberculosis possesses a great adaptive power, we did not think that the injection of living transmuted bacilli into man was allowable, harmless though it may seem.

Of the killed bacilli a vaccine is made in the ordinary way, which was called transmutan to avoid the ominous word tuberculosis, and now consists of the 7 strains at our disposal. In order to obtain a stock-transmutan, rightly called polyvalent, more and more transmuted strains will be taken up in the vaccine. In many lingering cases, it is quite possible to

employ auto-transmutan, which has so far been tried in only two cases.

Two years ago the first trial was made with transmutan, to which SPRONCK invited some of his ex-pupils, who acquitted themselves so well of this task, that their results of the first year published by Dr. ENKLAAR [1]), were confirmed further in the second year, in a still higher degree. While in the first year 9 colleagues made experiments, in the second year 128 partook of the experiments out of their own impulse.

Also in the second year we chiefly tried transmutan in frequently occurring pulmonary tuberculosis, although it would be difficult to name a second disease, in which the trial of a new remedy is more difficult. In all countries KOCH's tuberculin has been tried again and again for years, with the result that some consider the tuberculin therapy useful, and others call it useless. The difficulty in judging of a new remedy for pulmonary tuberculosis arises from the following circumstances. The disease offers a great variety. The anatomical lesions may be of small extent, yet of a serious character; they may also be less dangerous in spite of large extent. Moreover tuberculosis of the lungs is rather apt to spontaneous cure, and its course is so capricious that the ablest specialist is absolutely at a loss to make a prognosis.

It is evident that the activity of transmutan would have to be very great to cause an agreement concerning its utility in this disease in a few years.

New experience confirms the harmlessness of the vaccine. If the blood should contain too many antigens, as occurs in acute and far-advanced, lingering cases or with high fever, no good can be expected from transmutan, which even may be pronouncedly inimical.

The experiences of the second year also show that in not a few cases the sufferers manifest an improvement during the treatment, which with the application of other means had been expected for a long time in vain. This improvement consisted in subjective and objective recuperation and in clinical cure with or without a return of the power to work. As a rule recuperation or recovery was obtained only after long-continued treatment, but sometimes also surprisingly quick.

The patients that have improved during the vaccine-treatment or have been cured clinically have no advantage over those that are cured spontaneously. As might be expected a priori, a relapse may also occur in the treated patient.

In the case of recidivation it has struck us that when the treatment was resumed, improvement again was noticeable. In a more severe case of pulmonary tuberculosis the general condition improved considerably during the vaccine-treatment and the symptoms of the disease disappeared almost completely. But in the morning still some sputum was ejected occasionally that appeared to contain tubercle bacilli. Contrary to the advice of the physician the treatment was discontinued, while the patient got to his work again. A few months later again symptoms appeared, which induced him to

[1]) Geneeskundige Bladen, XXV, n⁰. 4, 1926.

stop his occupation. Immediately the vaccine-treatment was resumed and again improvement set in just as the first time.

The same observation was made in abdominal tuberculosis. During the treatment the subjective and the objective symptoms disappeared, and the treatment was discontinued because clinically the patient seemed to be restored. Some time later a relapse followed. After another vaccine-treatment the patient got better again. In these cases the treatment concerned outpatients, so that additional favourable circumstances such as bed-rest, better nutrition or attendance did not come into play.

Most often the cases of pulmonary tuberculosis did not belong to the light forms but to the severer ones. In the latter, however, transmutan has been tried, in the hope to save the sufferer, but sometimes it was too late. Anyhow, the favourable impression of a large number of physicians was not at all attained in the treatment of light, incipient cases, in which it is a pity that transmutan has hardly been tried.

In cases of abdominal tuberculosis very favourable results have been obtained and the rapid improvement was striking. Little is to be said about surgical cases, for lack of experience. But the number of surgeons, who try transmutan, increases so that after another twelvemonth they will have in one way or other a settled opinion about it.

In order to judge of the use of transmutan the most suitable experiments can be made in tuberculosis of the skin and of the mucous membranes of nose, mouth and pharynx, because the result can be verified so well. We are, therefore, highly pleased with Prof. VAN DER VALK's interest in the subject, who since November of last year has applied transmutan in his clinic and polyclinic at Groningen in a number of cases of lupus and other forms of tuberculosis cutis, and it was gratifying to learn that he has held a demonstration of patients, who had improved so much and so quickly by the administration of transmutan, that many colleagues were induced to ask us for it. The results of his examination, which is still going on, will no doubt be published in due time bij Prof. VAN DER VALK. We are also happy to state that Prof. BENJAMINS of Groningen has put this inquiry in hand in his clinic, and we are not less pleased to say that he obtained already some positive results, which encouraged him to further experimentation.

Several colleagues observed that during the transmutan-treatment a tuberculous process can not only be checked, but that also existing anatomic, tuberculous lesions can recover. Regression of the anatomic changes has also been observed in tuberculous processes in lungs, abdomen, etc., but cases of tuberculosis of the skin and the mucous membranes are to be considered in the first place as suitable to establish regression with certainty, and to make out whether complete cure can be attained.

An indicium would be very convenient to prevent deceptions in the transmutan-treatment. In cases of pulmonary tuberculosis a division into light,

middle severe, and vere severe (fatal) cases is no doubt of importance, and a priori the best results of transmutan can be expected from light cases. But experiences like the following, show that it is difficult to foretell anything. In an establishment transmutan would be applied in four cases of tuberculosis of the lungs, three of which were rather severe cases, and one so severe that a good result was not expected. But in view of the harmlessness of the vaccine, the experiment was made also in this case to prevent the discouraging influence of exclusion. The result was that the four patients improved and contrary to our expectation the severest improved most.

It is obvious to suppose that the best results of transmutan are yielded by patients, possessing the property of being good antibody-producers. Now some observations make us suspect that the agglutination-titre of the blood-serum of the sufferer might give us an indication in this respect. In this connection we do not think of an influence of the agglutinins, but of the probability that the agglutination-titre is an index of the quantity of other antibodies attacking the parasite, that are present in the blood of the sufferer, for experience taught us that with bacterial infections and also with artificial immunization, the various antibodies increase at about the same rate.

If a tuberculous process is on the way to spontaneous recovery or was cured spontaneously a short time ago, the agglutination-titre seems to be high. Examples of this are the cases of abdominal tuberculosis mentioned in Table IV, nr. 4 (titre 1 : 200) and nr. 5 (titre 1 : 300); again in Table II the case of tuberculous coxitis cured spontaneously nr. 13 (titre 1 : 200) and the spontaneously recovering case of tuberculosis of the hand, nr. 18 (titre 1 : 300).

That in the stage of recovery we find a higher agglutination-titre, that is at a moment when the circulation of a large quantity of active antibodies in the blood can be assumed, favours the supposition that also in tuberculosis the production of the different antibodies goes on at the same rate, so that the agglutination-titre in connection with the clinical observation, is an indicium whether the patient is to be regarded as a good or a less good producer of antibodies.

Therefore, we are now ascertaining whether the transmutan-treatment of patients, whose blood-serum has a higher amount of specific agglutinins already before the treatment, yields a better result than the treatment of patients with a lower titre.

Clinical observations enable us already now to interpret low titres as follows : If the disease is running an acute course, we may assume too many antigens in the blood, which the organism cannot absorb. If the lingering disease is already far advanced, reduced production as well as large consumption of antibodies (adsorption by antigens) may be the cause of the low titre. In both cases the low titre is considered as a warning that the administration of transmutan is hazardous. In all other cases a low titre is regarded as an index that the whole organism does not or little partake of

the strife against the local process, or that the sufferer is less capable of producing antibodies. In the first case the transmutan-treatment may be deemed rational. In the second case it is still being tried at haphazard.

That the results became gradually better, may perhaps be attributed to gradual change of the vaccine, which originally was prepared from one transmuted strain and now from seven strains. Doubts are also entertained as to the duration of the activity of the vaccine. Since a few months back it has been recommended not to order the whole series of increasing doses at once but only 6 at a time, that the vaccine might be applied as fresh as possible. In order to have fresh transmutan at our disposal the preparation takes place at intervals of a few weeks and the amount left of the preceding preparations is destroyed.

SUMMARY.

1. From 7 strains of typical, virulent tubercle bacilli 7 strains of non-acidfast, avirulent tubercle bacilli have been cultivated, which may be called indentic.

2. As explanation it has not been assumed, that we have at length succeeded in isolating the long known non-acidfast rods, present in every culture of the typical tubercle bacillus, from the acid-fast rods, but that the parasite of tuberculosis, adapting itself to a poor existence, saves itself by transmutation.

3. The transmutant thrives well on the poor medium, on which the typical parasite does not propagate itself. Its resistance and stability are greater, so that its survival is more safeguarded than that of the typical parasite.

4. In the blood-serum of sufferers from tuberculosis specific agglutinins are demonstrable by means of the transmutant.

5. The significance of this simple agglutination-test for the diagnosis of tuberculosis is getting more and more obvious.

6. The agglutination-test begins to be useful also in estimating the specific resisting power of the tuberculous patient.

7. The favourable impression of the vaccinotherapeutic application of the transmutant, obtained in the first year, was confirmed and increased in the second.

8. The utility of this therapy is attributed to strengthening of the antitoxic-anti-infectious component of the immunity against tuberculosis.

Physics. — *Isotherms of monatomic substances and their binary mixtures.* XXVI. *Isotherms of helium at —183.0 and —201.5° C. and pressures of 3 to 8 atmospheres.* By G. P. NIJHOFF and W. H. KEESOM. (Comm. N⁰. 188*b* from the Physical Laboratory at Leiden.)

(Communicated at the meeting of October 29, 1927).

§ 1. *Introduction.*

In a former communication [1]) we pointed to the importance of measurements of gases in slightly compressed states in order to obtain values as exact as possible of the first virial coefficients. Especially for helium this is of great importance with a view to thermometry at low temperatures. Moreover, as we then also remarked already, the now examined pressure region for helium with its low critical pressure, is of particular importance, because comparison, by means of the principle of the corresponding states, becomes possible with other gases, for which the equation of state in the corresponding pressure region has already been exactly examined.

We found it a great difficulty that the divided manometer constructed by us for pressures of 1 to 4 atmospheres, which, as has also been described in the former paper, must be read with a cathetometer, is not very fit for a fast use, so that the measurements with it want comparatively much time. With a view to this we have constructed two closed manometers, which are being shortly described here.

§ 2. *The closed manometers M 6 and M 20.*

These have been arranged in the same way as those described by KAMERLINGH ONNES and HYNDMAN [2]).

The first, destined for pressures between 2 and 6 atmospheres, consists of a glass stem 150 cm long, with a section of about 0.08 cm^2 and a capacity of about 12 cm^3. To this was blown at the upper end a small reservoir of 6 cm^3 and at the lower end a reservoir of 18 cm^3. So if we read the top of the mercury meniscus to $^1/_{10}$ mm, when the mercury during the measurements is standing quite at the upper end in the stem we have still an accuracy of $^1/_{7500}$, whereas at a lower place of the mercury the relative accuracy increases proportionally to the volume.

[1]) G. P. NIJHOFF and W. H. KEESOM, These Proc. **28**, 963, 1925, Comm. Leiden N⁰. 179*b*.

[2]) H. KAMERLINGH ONNES and H. H. F. HYNDMAN, These Proc. **4**, 776, 1902, Comm. Leiden N⁰. 78*c*.

The second manometer in principle quite identical to the first has the following dimensions : large reservoir 51.3 cm^3, stem 7 cm^3 with a length of 143 cm, small reservoir 2.9 cm^3.

It must still be observed that it did not seem desirable to us to join the manometer for the reading of the lowest pressures to the piezometer by means of compressed air as is usually done for the other manometers at Leiden. Piezometer and manometer were being filled out of the same mercury reservoir and formed a couple of communicating vessels. In order to calculate the pressure in the piezometer, we must of course make corrections for the difference in height of the two mercury columns and for a possible temperature difference of the two water jackets.

§ 3. *The measurements and results.*

The measurements are made in the same way as indicated in our former communication [1]).

We were very pleased that, for a great part of these, Prof. BORIS ILIIN from Moscow was willing to work with us. We have collected the results of these measurements in a separate communication [2]). The remaining ones we give in table I. In table II the values of B_A are given then, which we have calculated from our measurements and from those of ours and of Professor ILIIN.

For our calculations we took the value $\alpha_A = 0.0036618$ used until now

TABLE I.

θ °C.	p int. atm.	pv_A	d_A	O—C (pv_A)
— 183.07	3.2279	0.33112	9.748[5]	+0.00012
	4.0064	.33144	12.088	+ 7
	5.1853	.33197	15.620	+ 4
	7.3067	.33277	21.957	— 17
	8.1456	.33327	24.441	— 6
— 201.52	2.9347	.26323	11.149	—0.00004
	3.5627	.26351	13.520	— 5
	6.0023	.29471	22.669	+ 5
	6.5786	.26478	24.845	— 10

[1]) G. P. NIJHOFF and W. H. KEESOM, These Proc. 28, 963, 1925, Comm. Leiden N⁰. 179*b*.

[2]) See the following communication N⁰. 188*c*. These Proceedings 31, 408, 1928.

TABLE II.

Measurements of NIJHOFF, KEESOM and ILIIN			
θ 0° C.	$B_A \cdot 10^3$	θ 0° C.	$B_A \cdot 10^3$
— 103.29	+ 0.366	— 235.77	+ 0.0295
— 146.50	+ 0.256	— 249.80	— 0.0085
— 183.07	+ 0.158	— 252.57	— 0.0100
— 201.52	+ 0.120	— 255.84^5	— 0.0237
— 224.94	+ 0.059	— 258.99	— 0.0340
Measurements of BOKS and KAMERLINGH ONNES.			
+ 20	+ 0.550	— 103.64	+ 0.364
0	+ 0.523	— 142.02	+ 0.270
— 37.40	+ 0.480	— 183.34	+ 0.185
— 70.30	+ 0.428		
Measurements of HOLBORN and OTTO.			
0	+ 0.5282	— 183.0	+ 0.1562
— 50	+ 0.4344	— 208.0	+ 0.0998
— 100	+ 0.3366	— 252.8	— 0.0093
— 150	+ 0.2295	— 258.0	— 0.0337
— 183.0	+ 0.1537		

at Leiden. We have given up the idea of a recalculation according to the determination of the fundamental pressure coefficient of helium by KEESOM and Miss VAN DER HORST [1]) since the values of B_A will only slightly change. But the values of $B = \dfrac{B_A}{A_A}$ will be changed somewhat. It is also for this reason that we have preferred to give the values of B_A in this communication, instead of those of B.

At the same time we herewith give the values of B_A, which were calculated by us from the measurements of BOKS and KAMERLINGH ONNES [2]) and for comparison the values of B_A, reduced to Leiden units, which have been found in Berlin [3]).

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1) W. H. KEESOM and Miss H. VAN DER HORST, These Proc. **30**, 970, 1927, Comm. Leiden N⁰. 188*a*.

2) J. D. A. BOKS and H. KAMERLINGH ONNES, Comm. Leiden N⁰. 170*a*.

3) L. HOLBORN and J. OTTO, Zs. f. Phys. **30**, 320, 1924 and **37**, 359, 1926.

These different values of B_A are represented in fig. 1.

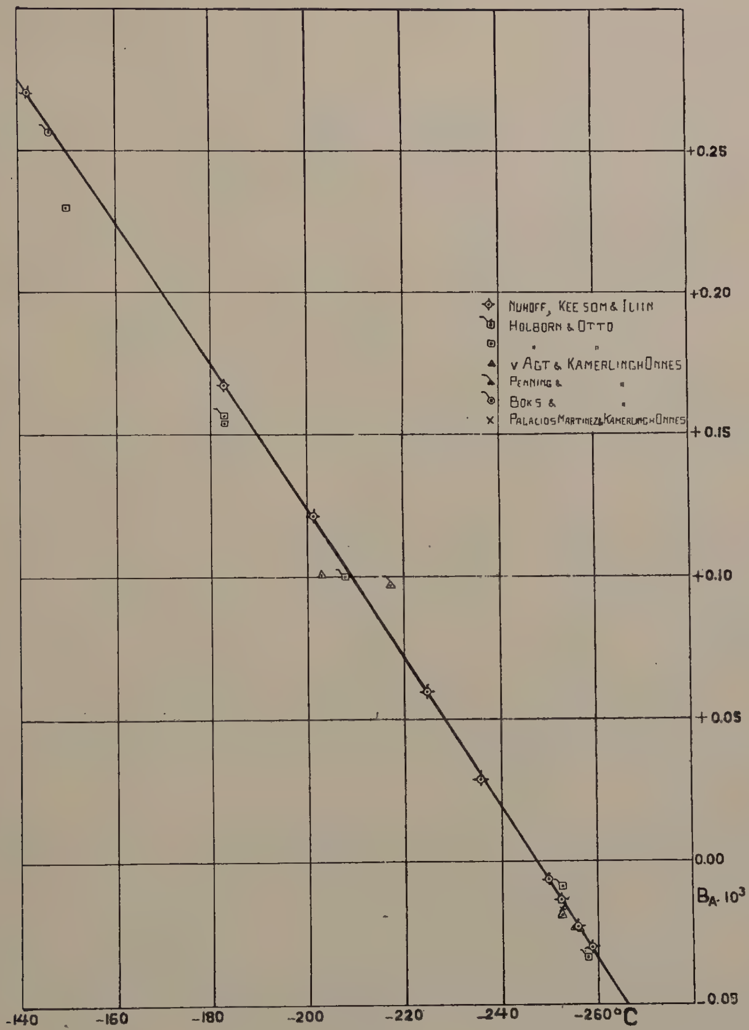

Fig. 1.

Herewith we observe that the German results, as far as the temperature region, which still can be reached with liquid oxygen, are lying lower than ours, while at the temperatures of liquid hydrogen, though the dispersion of the Berlin values is larger, the agreement can yet be called very satisfactory.

Besides it is very remarkable that at lower temperatures the B_A-values tend to depend linearly from the temperature and that after extrapolation this line passes through values found in the liquid helium region.

Finally we thank Miss A. SOLLEWIJN GELPKE for her help as well at the measurements, as especially at the calibration of the manometers.

Physics. — *Isotherms of monatomic substances and their binary mixtures,* XXVII. *Isotherms of helium between —103.6° C. and —259.0° C. and at pressures of 1.5 to 14 atmospheres.* By G. P. NIJHOFF, W. H. KEESOM and B. ILIIN. (Comm. N⁰. 188*c* from the Physical Laboratory at Leiden.)

(Communicated at the meeting of October 29, 1927).

We have measured isotherms of helium in the same way as has been described in the communication about oxygen [1]) with the same arrangement of the piezometer and the manometers.

The temperatures were obtained with the aid of liquid ethylene, liquid hydrogen and with the hydrogen vapour cryostat.

The piezometer with which the volumes were measured consisted of the same stem of 108 cm³, with which also the oxygen isotherms have been measured; however, with a view to the so much lower temperatures at which we wished to measure, the gas reservoir had been replaced by a larger one of 400 cm³ so that the normal volume with which we worked, amounted to about 500 cm³. The small reservoir in the cryostat had a capacity of 20 cm³, so that our greatest density amounts to about 25 Amagat-units.

The following table gives the values found by us. The B_A's, which can be calculated from these, have been published in the preceding communication [2]). The last column gives the differences between the observed values of pv_A, and those which have been calculated with the just mentioned B_A's.

As could also be expected from an estimation of that term using the reduced virial coefficients VII 1 [3]), it appears from the column O—C that the term with C_A in the development of pv_A to ascending powers of v_A^{-1} does not come into account in the region of temperatures and pressures treated here. It is evident that this benefits the exactness of the determination of B_A [4]).

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1) G. P. NIJHOFF and W. H. KEESOM, These Proc. **28**, 963, 1925, Comm. Leiden N⁰. 179*b*.

2) G. P. NIJHOFF and W. H. KEESOM, These Proc., page 404; Comm. Leiden N⁰. 188*b*.

3) H. KAMERLINGH ONNES and W. H. KEESOM, Comm. Leiden Suppl. N⁰. 23, § 36.

4) For the interest which exact measurements of the isotherms of gases and in connection with this, the determination of the attraction quantity of VAN DER WAALS have for the theory of absorption, compare B. ILIIN, Phil. Mag. (6) **48**, 193. 1924. Zs. f. Phys. **33**, 435, 1925.

TABLE I.

θ °C.	p int. atm.	pv_A	d_A	O—C (pv_A)
—103.30	14.242	0.62988	22.610	+0.0002
	9.910	.62743	15.792	+ 2
— 146.62	9.6830	.46792	20.694	
— 224.94	4.4156	.17769	24.848	0.00000
	4.1425	.17759^5	23.326	0
	2.7909	.17721	15.747	+ 6
— 235.77	3.4229^5	.13737	24.966	+0.00004
	2.9017	.13720	21.150	— 1
	2.4784	.13713	18.074	+ 1
	2.1585	.13709	15.745	+ 4
— 249.80	2.1315^5	.085008	25.075	+0.00002
	2.0766	.085017	24.426	+ 1
	1.9183	.085027	22.561	+ 2
	1.7267	.085035	20.308	— 3^5
— 252.57	1.8842	.074865	25.168^5	+0.00002
	1.8816	.074865	25.133	+ 2
	1.8373	.074865	24.542	+ 1
	1.4710	.074882^5	19.644^5	— 2
— 255.84^5	1.5793	.062504^5	25.249	—0.00001
	1.5430	.062533	24.661	0
	1.4611	.062565	23.326	+ 0^5
	1.3109^5	.062673	20.299	+ 4
— 258.99	1.28305	.050756	25.279	+0.00001
	1.12572	.050850^5	22.075	0

Physics. — *Isotherms of di-atomic substances and their binary mixtures.* XXXIV. *Isotherms of hydrogen at temperatures of 0° C. and +100° C.* By G. P. NIJHOFF and W. H. KEESOM. (Comm. N⁰. 188*d* from the Physical Laboratory at Leiden.)

(Communicated at the meeting of December 17, 1927).

The isotherms of +100° C. and 0° C. have been measured with a piezometer with a capacity of 1500 cm^3, the same with which VAN URK and KAMERLINGH ONNES [1]) have measured part of their nitrogen isotherms. To this we connected a reservoir with a capacity of 33.6 cm^3. The pressures were measured with the aid of the closed manometer M 60. The temperature of 0° C. was obtained with the aid of finely planed ice made from water of the main, whereas for that of + 100° C. the steam apparatus described by KAMERLINGH ONNES in Comm. N⁰. 27 [2]) was used, still in the old shape [3]). The temperature of the vapour in this apparatus was measured with a BECKMANN-thermometer of which the steam point had been determined separately.

Beforehand, for the sake of control, we have first measured three points at +20° C., which agreed well with SCHALKWIJK's isotherm. In the last column of table I we give the differences between the observed pv_A's and the values of pv_A calculated with the aid of values of B_A and C_A, which we communicate in table II and which, in order to make them correspond as well as possible, we chose somewhat differing from the values of SCHALKWIJK [4]) and from the later ones of KAMERLINGH ONNES, CROMMELIN and Miss SMID [5]).

Concerning the isotherms of +100° C. and 0° C., the region of pressures in which we have measured does not seem to be too favourable for the determination of the values of B; for the C plays a not to be neglected part here.

For 100° C. we determined B_A and C_A as follows. HOLBORN and OTTO [6]), who have measured to about 100 atmospheres, give for +100° C. in their development according to ascending powers of the pressure, only a second term, whereas they don't want a quadratic term. If we compare

[1]) A. TH. VAN URK and H. KAMERLINGH ONNES, Comm. Leiden N⁰. 169*d*. For the calibration see also A. TH. VAN URK, Thesis Leiden.

[2]) H. KAMERLINGH ONNES, Verslagen Kon. Ak. v. Wet. Amsterdam **5**, 79, 1896, Comm. Leiden N⁰. 27.

[3]) Compare W. H. KEESOM and Miss H. v. D. HORST, These Proc. **30**, 970, 1927; Comm. Leiden N⁰. 188*a*.

[4]) J. C. SCHALKWIJK, These Proc. **4**, 107, 1902; Comm. Leiden N⁰. 70.

[5]) H. KAMERLINGH ONNES, C. A. CROMMELIN and Miss E. I. SMID, These Proc. **18**, 465, 1915, Comm. Leiden N⁰. 146*b*.

[6]) L. HOLBORN and J. OTTO, Zs. f. Physik. **33**, 1, 1925.

their development according to p with a development in series according to d_A, then from the coefficients of the first development in series, we can calculate the corresponding coefficients of the last mentioned series. In this way we calculated the value of B_A and C_A communicated in table II. In the last column of table I we give the differences between the observed values of pv_A and the values, which we calculated with the values just mentioned of B_A and C_A. We conclude from these that the values of B_A and C_A derived from HOLBORN and OTTO give sufficient correspondence also for our measurements.

As the value $C_A = 0.606 \times 10^{-6}$ calculated from the measurements of AMAGAT for the same temperature, is only little different, we can also put some trust into the value used by us $C_A = 0.635 \times 10^{-6}$.

For the isotherm of 0° C. we find the best correspondence with $B_A = 0.605 \times 10^{-6}$ and $C_A = 0.565 \times 10^{-6}$, whereas from the measurements of HOLBORN and OTTO follows $B_A = 0.620 \times 10^{-3}$ and $C_A = 0.760 \times 10^{-6}$, and from AMAGAT has been calculated for $C_A = 0.670 \times 10^{-6}$.

The measured quantities are following here:

TABLE I.

θ °C.	p int. atm.	pv_A	d_A	O—C (pv_A)
+20	32.006^5	1.0947	31.063^5	−0.0003
	40.098	1.0990	36.485	0
	45.771^5	1.1021	42.527	− 6
+100	39.964	1.3929	28.691	− 2
	43.852	1.3951	31.431^5	+ 3
	48.746	1.3987	34.850	− 0^5
	54.603	1.4028	38.924	0
	59.391	1.4061	42.237	+ 1
0	32.312^5	1.0188	31.715	-0.0003^5
	32.323	1.0190	31.721	− 1^5
	33.524	1.0194	32.885	− 5
	34.875	1.0206	34.171	− 1
	36.306^5	1.0217	35.536	+ 1
	37.883	1.0226	37.047	0
	39.545^5	1.0231	38.652	− 5
	42.905^5	1.0260	41.817	+ 3
	44.085	1.0264^5	42.949	+ 0^5
	44.119	1.0266	43.284	0

In the following table we collect the values of B_A and C_A found by the different observers.

TABLE II.

	$B_A \cdot 10^3$	$C_A \cdot 10^6$
0° C.		
AMAGAT	0.669	0.670
KAMERLINGH ONNES and BRAAK	0.580	0.670
WITKOWSKI	0.619	
CHAPPUIS	0.605	
HOLBORN and OTTO	0.620	0.760
VERSCHOYLE	0.626	0.560
NIJHOFF and KEESOM	0.605	0.565
20° C.		
SCHALKWIJK	0.667	0.993
KAMERLINGH ONNES, CROMMELIN and Miss SMID	0.657	1.119
VERSCHOYLE	0.698	0.533
NIJHOFF and KEESOM	0.677	0.797
100° C.		
AMAGAT	1.057	0.606
WITKOWSKI	0.920	
HOLBORN and OTTO	0.937	0.635
KAMERLINGH ONNES and BRAAK	0.863	0.606
NIJHOFF and KEESOM	0.937	0.635

In fig. 1 the most important values of B_A for this temperature region have been indicated.

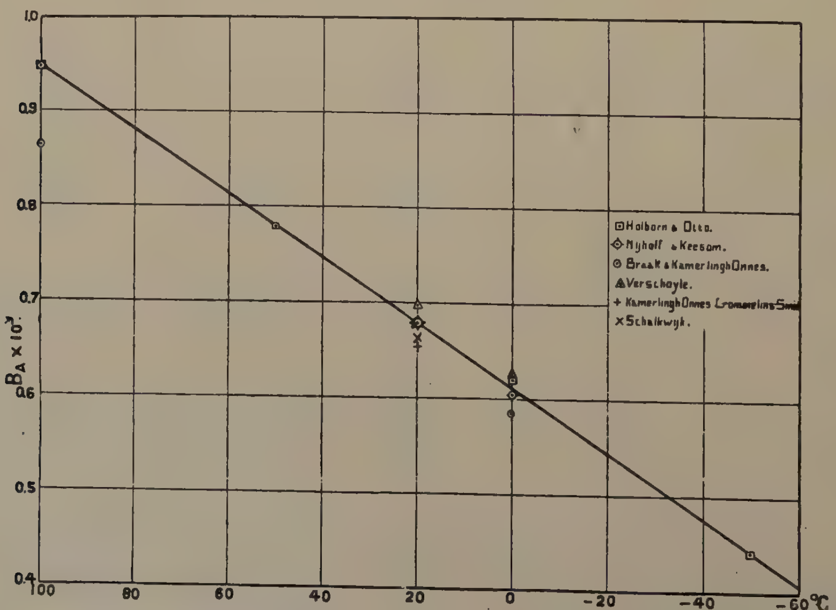

Fig. 1.

Physics. — *Isotherms of di-atomic substances and their binary mixtures.* XXXV. *Isotherms of hydrogen at temperatures of —225.5° C. to —248.3° C. and pressures of 1.6 to 4.2 atmospheres.* By G. P. NIJHOFF and W. H. KEESOM. (Comm. N⁰. 188e from the Physical Laboratory at Leiden.)

(Communicated at the meeting of January 28, 1928).

In the same way as has been indicated in a preceding communication [1]) we have measured isotherms of hydrogen at temperatures of —225.5° C. to —248.3° C. The piezometer used for this purpose was the same as the one used in the previous case, while the pressures were measured with the closed manometer as was also described in Communication N⁰. 188*b*. The temperatures were obtained by the hydrogen-vapour-cryostat.

From the measurements which are given below, we have found the following values for B_A:

θ ° C.	$B_A \cdot 10^3$
—225.54	—0.268
—231.52	.309
—236.56	.340
—241.84	.390
—248.32	.439

The deviations of the observed values of pv_A, calculated with the aid of the given B_A-s are added in the last column of the following table.

The values of B_A mentioned here, agree well with those which VAN AGT and KAMERLINGH ONNES [2]) have measured with the aid of the thermometer of constant volume and variable density at temperatures of liquid hydrogen.

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1) G. P. NIJHOFF and W. H. KEESOM, These Proceedings p. 404, 1928, Comm. Leiden N⁰. 188*b*.

2) F. P. G. A. J. VAN AGT and H. KAMERLINGH ONNES. These Proceedings **28**, 614, 1925, Comm. Leiden N⁰. 176*b*.

θ °C.	p int. atm.	pv_A	d_A	O—C (pv_A)
—225.54	2.7313	0.16982	16.086	+ 0.00012
	3.4419	.16855	20.417	+ 1
	3.9391	.16772	23.481	0
	4.1727	.16727	24.950	— 5
—231.52	2.4830	0.14676	16.915	— 0.00010
	2.8937	.14593	19.830	— 5
	3.4443	.14477	23.786	+ 1
	3.6201	.14440	25.063	+ 3
—236.56	2.0808	0.12808^5	16.246	— 0.00008
	2.4734	.12695	19.483	— 11
	2.7297	.12628	21.616	— 5
	3.0178	.12565	24.001	+ 13
	3.1453	.12523	25.101	+ 8
—241.84	2.5531	0.10487	24.346	0.00000
	1.7959	.10773	16.665	— 13
	2.0727	.10682	19.399	+ 3
	2.3408	.10577	22.121	+ 4
—248.32	1.6290	0.08185	19.925	— 0.00005
	1.7349	.08128	21.303	— 2
	1.9614	.07984	24.581	— 1
	2.0078	.07968	25.209	+ 10

Geology. — *The potential energy of the gas in the oil bearing formations.* By J. VERSLUYS.

(Communicated at the meeting of May 26, 1928).

Mineral oil accumulated in pools in the earth is as a rule saturated with gas at the prevailing pressure and in many cases some free gas is still found in the highest parts of the oil bearing anticlines and domes. If in some cases the oil was not saturated with gas at the initial pressure, the pressure would decline in the vicinity of the borehole as soon as the oil bearing stratum was struck and some oil and gas had escaped. On account of this some gas would be set free in the vicinity of the borehole and even the pressure of the edgewater would not re-establish the original pressure near the borehole, until all liberated gas had been dissolved again. So for some time the pressure observed would be approximately equal to that at which the oil would be saturated with the absorbed gas and this would be the pressure measured in the borehole.

If an oil has absorbed gas and afterwards the pressure is reduced a part of this gas can be set free and expand after it has been set free. Owing to this the gas is able to exert energy and this energy is supposed to yield the principal force expelling the oil from the porous rocks into the wells.

The object of this paper will be to find a mathematical expression for the extent of the potential energy present in this form. The work performed by the gas if it is set free and expands is supplied by the molecular energy and cooling would be the result. Assuming that the heat needed to restablish the initial temperature is supplied immediately we may accept that the process is isotherm and further we will disregard all deviations from the laws of BOYLE and HENRY.

Should a volume of oil q be under a pressure of p atmospheres and should it be saturated with gas at that pressure, the coefficient of absorption being a, that volume of oil would contain a quantity of gas, which would occupy at atmospheric pressure a volume

$$a q p . \qquad (1)$$

If the pressure declines by dp a certain quantity of gas would be set free, occupying at the atmospheric pressure a volume

$$aq\, dp \qquad (2)$$

and at the pressure p under which the oil and gas are:

$$aq \frac{dp}{p} \qquad (3)$$

The volume of the oil and the gas associated with it, being q at the beginning, is increased by the volume expressed under (3). The work performed by the gas is then :

$$dA_1 = a\,\alpha q\,dp, \quad \ldots \ldots \ldots \quad (4)$$

if the pressure of one atmosphere equals a units of power per unit of area.

If the pressure declines from P_2 at which the oil is saturated with the gas it contains, to a pressure P_1, in this manner, i.e. at being set free, the gas will perform an amount of work :

$$A_1 = a\,\alpha q\,(P_2 - P_1) \quad \ldots \ldots \ldots \quad (5)$$

As the pressure decreases between those limits P_2 and P_1 the gas liberated while the pressure declined from p to $p—dp$ $(P_2 > p > p—dp > P_1)$, will still expand owing to the pressure declining from $p—dp$ to P_1. The work performed by the quantity of gas, which would occupy the unit of volume at atmospheric pressure, should the pressure decline from p_2 to p_1 is:

$$a \log \frac{P_2}{P_1} \quad \ldots \ldots \ldots \quad (6)$$

according to a familiar formula.

Hence the quantity of gas set free between the limits of pressure p and $p—dp$ would, by expansion owing to the decline of pressure from $p—dp$ to P_2, perform work :

$$dA_2 = a\,\alpha q \log \frac{p}{P_1}\,dp \quad \ldots \ldots \ldots \quad (7)$$

The work performed by the expansion of the gas set free if the pressure declines from P_2 to P_1 is :

$$A_2 = a\,\alpha q \left\{ \int_{P_1}^{P_2} \log p\,dp - \log P_1 \int_{P_1}^{P_2} dp \right\} = a\,\alpha q \left\{ P_2 \log \frac{P_2}{P_1} - (P_2 - P_1) \right\}. \quad (8)$$

Hence, the total energy exerted by the gas, if the pressure declines from P_2 at which the oil is saturated to a smaller pressure P_1, is :

$$A = A_1 + A_2 = a\,\alpha q\,P_2 \log \frac{P_2}{P_1} \quad \ldots \ldots \ldots \quad (9)$$

The product $\alpha q\,P_2$ in this equation is the volume which would be occupied by all the gas originally absorbed in the oil at atmospheric pressure. If this volume be put at

$$\alpha q\,P_2 = Q_2 \quad \ldots \ldots \ldots \quad (10)$$

(9) becomes converted into :

$$A = a\,Q_2 \log \frac{P_2}{P_1} \quad \ldots \ldots \ldots \quad (11)$$

A represents the energy which the gas being set free from the oil between

the limits of pressure P_2 and P_1 exerts in two manners; viz. owing to the liberation and owing to the expansion. We must keep in mind that the gas still remaining absorbed in the oil would at atmospheric pressure occupy a volume

$$Q_1 = aq\, P_1 \qquad (12)$$

and the gas set free between the limits of the pressure P_2 and P_1, would at atmospheric pressure occupy a volume:

$$aq\,(P_2 - P_1) = Q_2 - Q_1 \qquad (13)$$

Hence the energy expressed by (11) is exerted by a quantity of gas $Q_2 - Q_1$ in the two manners exposed above.

But the work expressed by (11) also equals the work which would be performed by a quantity of free gas Q_2 in expanding between the same limits of pressure P_2 and P_1 according to (6).

So we have deduced: if a volume of oil q at the pressure P_2 and P_1 ($P_2 > P_1$) would be saturated by quantities of gas occupying volumes respectively Q_2 and Q_1, at atmospheric pressure, the volume of gas set free if the pressure after saturation declined from P_2 to P_1, would occupy a volume $Q_2 - Q_1$ at atmospheric pressure and this quantity of gas would during this process perform work equal to that performed by all the gas (Q_2) associated with the oil if it were free and expanded between the same limits of the pressure P_2 and P_1, this work being expressed by (11).

If the oil is — which often is the case in the highest parts of the structure — at the pressure P_1 not only associated with the quantity of gas Q_2, saturating it at this pressure, but also with a quantity of free gas, occupying a volume Q_f at the atmospheric pressure, this free gas would if the pressure declined from P_2 to P_1 exert an energy

$$a\, Q_f \log \frac{P_2}{P_1} \qquad (14)$$

and the total work performed by the quantities of gas Q_2 and Q_1 would be

$$a\,(Q_2 + Q_b) \log \frac{P_2}{P_1} \qquad (15)$$

So in general for the work performed by all the gas (absorbed and free) associated with the oil in a pool if the *reservoir pressure ("rock-pressure")* P_r be smaller than or equal to the pressure at which the oil would be saturated with the gas present, may be written

$$aQ \log \frac{P_r}{P_o}, \qquad (16)$$

if P_o represents the pressure under which the oil and the gas leave the formation and Q the volume, all the gas, free and absorbed in the oil would occupy at atmospheric pressure.

Geology. — *Determination of the pressure in gas containing strata.* By J. VERSLUYS.

(Communicated at the meeting of May 26, 1928).

The pressure in the oil bearing strata is a factor of importance in oil recovery, because from this pressure the quantity of gas dissolved in the oil can be calculated. In a porous rock containing liquid without gas the pressure can be computed from the specific gravity of the liquid and the level to which it rises in a well. The pressure in strata holding only gas may be deduced from the pressure of the gas in a closed well near the surface.

Generally it is accepted that the pressure exerted by the weight of the gas in the well may be neglected which would mean that the pressure at the bottom would be the same as the pressure at the head of the well. This however is not by any means the case, in deep wells great pressures occur. The specific gravity of the gas under a great pressure may not always be neglected. The initial pressure of the gases and fluids in the pores of the rocks as a rule does not differ much from the hydrostatic pressure of a column of water which would fill the borehole up to the surface. This was stated i.a. by J. H. GARDNER (III) in 1916, the phenomenon has since been observed in many districts and is mentioned by several authors.

According to this observation, in a sand at a depth of 1000 metres the pressure would be about 100 atmospheres. With this pressure a gas, the specific gravity of which in grams/cub. centimeter is 0.001 at atmospheric pressure, would have a specific gravity of $^1/_{10}$ of that of water. If 100 atmospheres pressure are measured at the head of a closed well the weight of the column of gas would equal that of a column of water of 100 metres, so there must be added about 10 atm. to the pressure at the head of the well and the gas pressure in the sand would be about 110 atmospheres. These figures are arbitrarily chosen, but it is clear that the pressure in the well-head can considerably differ from the pressure in the earth layer.

The correction to be applied to the measured gas pressure at the head of a well can be simply deduced. If the pressure at a height y above a certain level in the well is p atmospheres and the specific gravity of the gas γ, then the pressure exerted by the column of the gas enclosed by the levels y and $y + dy$ in units of power per unit of area is

$$p\gamma \, dy \qquad \ldots \ldots \ldots \ldots (1)$$

If the pressure of one atmosphere is equal to a units of power per unit of area, we may write:

$$p\gamma\, dy = -a\, dp \quad \ldots\ldots\ldots\ldots \quad (2)$$

or

$$dy = -\frac{a}{\gamma}\frac{dp}{p} \quad \ldots\ldots\ldots\ldots \quad (3)$$

Integrating this equation between the limits y_2 and y_1, respectively p_2 and p_1, we get:

$$y_2 - y_1 = -\frac{a}{\gamma} \log \frac{p_2}{p_1} \quad \ldots\ldots\ldots \quad (4)$$

and if y_1 and y_2 be taken for the heights of the gas-bearing stratum and the head of the well $y_2 - y_1 = D$ is the depth of the well. So

$$D = \frac{a}{\gamma} \log \frac{p_1}{p_2} \quad \ldots\ldots\ldots\ldots \quad (5)$$

or

$$p_1 = p_2\, e^{\frac{D\gamma}{a}} \quad \ldots\ldots\ldots\ldots \quad (6)$$

Hitherto p_2 and p_1 have been expressed in atmospheres; this formula (6), however, holds if any other unit of pressure be chosen.

If D be expressed in meters, then $a = 10.33$ and then the pressure p_1 in the gas-bearing sand would be computed from the formula:

$$p_1 = p_2\, e^{\frac{D\gamma}{10.33}} \quad \ldots\ldots\ldots\ldots \quad (7)$$

in which p_2 is the pressure in the closed well, measured at the surface, γ the specific gravity of the gas in grams per cubic centimeter and D the depth of the well in meters. If the depth be expressed in feet and the specific gravity in pounds per cubic foot this formula becomes

$$p_1 = p_2\, e^{\frac{D\gamma}{2093.6}} \quad \ldots\ldots\ldots\ldots \quad (8)$$

This pressure p_1 is generally called *"rockpressure"* which term however in recent times is being replaced by the more correct expression *"reservoir pressure"* (see I page 363 and II).

If we multiply the volume of a gas pool by its average porosity and the reservoir pressure p_1, we find the amount of gas contained in the pool disregarding phenomena of adsorption.

Should gas and liquid be associated in a porous bed, the reservoir pressure cannot be calculated from the pressure at the head of the closed well, this as a rule partly being filled with liquid. W. B. HEROY (I, p. 371) states that in literature many very low figures for the reservoir pressure are encountered, these figures practically giving the pressure read at the head of the closed well (the so-called closed pressure).

In W. B. HEROY's paper two methods are described of determining the reservoir pressure in oil-bearing strata. One of these methods (see I, p. 366) is to pump gas into the well, forcing the liquid from the well into the porous rock. The gas pressure required to produce this result is the pressure p_2 in (7) and (8) and the reservoir pressure is p_1 computed with the aid of one of these formulae.

Multiplying the volume of oil in the subterranean reservoir by the absorption coefficient and the reservoir pressure, according to the HENRY law, we get the amount of gas absorbed in the oil.

LITERATURE.

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Bacteriology. — *Von dem Stoffwechsel und der Verbreitung der Gärungssarcinen. (Sarc. ventriculi* GOODSIR *und Sarc. maxima* LINDNER.) By JAN SMIT. (Communicated by Prof. W. SCHÜFFNER.)

(Communicated at the meeting of March 31, 1928)

Durch die vortreffliche Untersuchung der Gärungssarcinen, die Herr Prof. BEIJERINCK zusammen mit Herrn Dr. GOSLINGS in den Jahren 1905 und 1911 [1]) verrichtete, wurde die damals ebenso neue als überraschende Tatsache festgestellt, dass man Kulturen der Magensarcinen (*Sarc. ventriculi* GOODSIR), deren Anwesenheit bei einigen Magenstörungen man schon seit 1842 kannte, die aber immer für unkultivierbar gegolten hatten, sehr leicht gewinnen kann, indem man von Gartenerde ausgeht, wobei die zuckerhaltige Nahrungsflüssigkeit (Malzwürze oder Bouillon mit Zucker) so stark angesäuert wird (mit 8 cc N. Phosphorsäure oder 6—7 cc N. Salzsäure pro 100 cc), dass sie nur noch für die in der Erde befindlichen Sarcinen geniessbar ist. Nach 24 Stunden bei 37° verrät sich ihr Wachtstum durch eine sehr starke Gasbildung, und das mikroskopische Präparat des Erdsatzes weist eine grosse Menge sehr grosser Sarcinenpakete auf, deren Bild vollkommen übereinstimmt mit dem der im Magen aufgefundenen Sarcinen.

Anderseits wurde von LINDNER [2]) und HENNEBERG [3]) auf die Tatsache hingewiesen, dass sich in spontan gegorenen Mehlbreien und in „buttersauren Maischen" bisweilen ebenfalls eine grosszellige Sarcine findet, die äusserlich stark der Magensarcine gleicht, von genannten Forschern aber nicht näher untersucht wurde, wahrscheinlich da sie sich nicht rein züchten liess.

In den genannten Veröffentlichungen BEIJERINCKS wird diese Sarcinenart mit denen aus Boden und Magen identifiziert, zwar ohne näheren Beweis. Es kam mir daher wichtig vor, diese Sarcinen dreifacher Herkunft einer eingehenderen, vergleichenden Untersuchung zu unterwerfen.

Es war dazu an erster Stelle notwendig, obengenannte Säuregrade der von BEIJERINCK benutzten Kulturflüssigkeiten auf eine besser zu vergleichende Basis zu stellen, durch die Bestimmung der Wasserstoffionenkonzentration, wobei der Anreicherungsversuch mit Erde am besten gelingt und auch durch Beobachtung der Widerstandsfähigkeit der Reinkultur der 3 Mikroben verschiedenen Säuren gegenüber.

Als Resultat dieser anderweitig [4]) veröffentlichten ausführlichen Unter-

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[1]) Siehe diese Proceedings, **7**, 580 (1905) und **13**, 1237 (1911).

[2]) LINDNER, Mikroskop. Betriebskontrolle in den Gärungsgewerben 3⁰ Ed. S. 342.

[3]) HENNEBERG, Gärungsbakt. Praktikum S. 100.

[4]) Siehe Ned. Tijdschr. v. Hygiene I, 201 (1927) und II, 210 (1927).

suchung erwähne ich hier nur, dass bei Anwendung von Malzwürze, angesäuert mit HCl bis zu einem p_H Wert von 1.3—1.5 (8—10 cc Norm. HCl pro 100 cc Würze) gewöhnlich die besten Erfolge erzielt werden, dass aber auch dann zahlreiche Erdmuster kein Wachstum der Sarcine veranlassen, welche Misserfolge bei Anwendung von Phosphorsäure (bis p_H = ca. 2—2.2 oder 10—15 cc N. Säure per 100 cc) noch bedeutend häufiger waren. Ausser den genannten Säuren waren auch Salpetersäure, Schwefelsäure und Milchsäure mehr oder weniger geeignet, während Essigsäure und Oxalsäure Misserfolge herbeiführten. Von einer so weitgehenden Widerstandsfähigkeit freier Salpetersäure gegenüber (p_H = 1.5 war ungefähr die Grenze) wird dies wohl das erste Beispiel sein.

Mit den Reinkulturen der Sarcinen liessen sich die Grenzen noch genauer bestimmen. Es zeigte sich, dass Ansäuerung mit Salzsäure ertragen wurde bis zu einem p_H von 0.8 (20 cc Norm. pro 100 cc), Salpetersäure bis zu 1.5, Milchsäure bis 2.9. Die Widerstandsfähigkeit gegen Phosphorsäure ist sehr gross: 70 cc N. per 100 cc wird noch ausgehalten, wobei aber das p_H nur bis auf 1.45 gefallen ist.

Auf der alkalischen Seite liegt die Grenze bei p_H = 9.8. Das Wachstumsgebiet liegt also innerhalb sehr weiten Grenzen.

Zwei Kennzeichen fanden sich, wodurch *S. maxima* sich von den beiden andern unterschied, die indertat in all ihren Eigenschaften sich als identisch herausstellten und daher unter dem Namen *S. ventriculi* zusammengefasst bleiben dürfen. Es waren: die Zellulose-reaktion mit Chlorzinkjodium, die für *S. ventriculi* positiv, für *S. maxima* negativ ausfällt, und der Stoffwechsel, der sehr grosse Unterschiede aufweist.

Schon BEIJERINCK hat *S. ventriculi* sehr richtig als obligate Zuckermikrobe beschrieben, welche die Eiweisse von Pepton, Bouillon, Hefenwasser, Malzextrakt u.ä. braucht, um wachsen zu können, daneben aber noch Zucker (Glukose, Laevulose, Saccharose, Laktose [1]), Maltose) erfordert. Sie ist aber nur zur Verarbeitung weniger Prozente Zucker, 1½ bis 2 %, imstande, sodass die meistens stürmisch einsetzende Gärung nach wenigen Tagen aufhört, während noch genügend Stickstoff vorrätig ist sowie aller Zucker, der über genannten Betrag hinaus hinzugefügt sein dürfte. Es zeigte sich, dass *S. maxima* noch weniger Zucker verarbeiten konnte: nur ½ bis 1 %. Ueber die Stoffwechselprodukte war aber so gut wie nichts bekannt. Ausser den beiden Gasen Kohlensäure und Wasserstoff (Verhältnis ihres Volumens ungefähr 3 : 1) findet man bei BEIJERINCK Milchsäure als Produkt erwähnt. Es zeigte sich mir aber, dass davon entweder nichts oder nur wenig entsteht, während dagegen ungefähr 10 % vom verschwundenen Zucker an Essigsäure festgestellt wurde. Die Hauptprodukte der Zuckerumwandlung von *S. ventriculi* sind aber Kohlensäure und Alkohol in Mengen, die sich der der Alkoholhefe nähern. Zufügung von Kreide hat

[1]) Dass BEIJERINCK diesen Zucker zu den nicht brauchbaren rechnet, muss auf einem Irrtum beruhen. Sowohl in Roh- als Reinkultur war die Vergärung der Laktose vollkommen normal.

auf die Mengenverhältnisse nur wenig Einfluss. Nur ist *S. maxima* imstande, 2 % Zucker zu verarbeiten.

Stoffwechsel der Sarcinen.

	S. ventriculi			S. maxima
Produkte (in Proz. des verschwundenen Zuckers)	Hefenwasser 2 % Glukose (verschwunden 1.52 %)	Hefenwasser 2 % Laevulose (verschwunden 1.94 %)	Peptonwasser 2 % Glukose (verschwunden 1.82 %)	Hefenwasser 2 % Glukose (verschwunden 0.6 %)
Kohlensäure	41.7 %	45.2 %	42.2 %	36.3 %
Wasserstoff	0.58	0.7	0.78	2.55
Aethylalkohol	40.3	50.6	40.2	Spore
Ameisensäure	1.08	—	2.4	1.05
Essigsäure	9.0	3.7	13.0	10.0
Buttersäure	—	—	—	37.1
Bernsteinsäure	—	—	—	3.44
Milchsäure	3.05	—	—	10.7
Acetylmethylkarbinol	Spore	0.06	1.2	—
Total	95.72 %	100.26 %	99.8 % [1]	101.14 %

In obiger Tabelle findet man die Zahlen vereinigt, auch für die Produkte von *S. maxima,* unter denen der Alkohol gänzlich fehlt, doch von einer grossen Menge Buttersäure ersetzt wird. Weiter findet sich auch Milchsäure unter den Produkten, während sich das Verhältnis der Kohlensäure zum Wasserstoff merklich verringert hatte (an Volumen 1 : 1.4), wodurch es sich zeigt, dass auch in dieser Hinsicht diese Sarcine den echten Buttersäurebakterien nahe steht (für *Gran. saccharobutyricum* findet DONKER [2]) Werte von ungefähr 1 : 1.2). Dagegen findet sich das Volumenverhältnis der Gase bei *S. ventriculi* gerade bei der fakult. anaeroben *Cl. polymyxa,* von der einige Stämme (u.a. DONKER's Stämme 11 und 4) auch weiter in ihrem Stoffwechsel einigermassen der Sarcine gleichen.

Oben erwähnte ziemlich zahlreiche Misserfolge der Anreicherungsversuche mit Erde führten mich zu einer nähern Untersuchung einer grossen Zahl von Erd-, Schlamm-, und Sandmustern aus verschiedenen Gegenden Hollands, nebst einigen Mustern getrockneter Erde, die mir das Kolonial-

[1]) Dank der Freundlichkeit von Herrn Prof. KLUYVER in Delft ist diese Analyse im dortigen Laboratorium mit dem grossen Gärungsapparat (60 L. Inhalt) geschehen, wofür ich ihm und seinen Assistenten, den Herrren Ing. VAN NIEL und LEEFLANG, zu Dank verpflichtet bin.

[2]) H. J. DONKER, Dissertation, Delft 1926.

institut in Amsterdam freundlichst überliess. Aus dieser Untersuchung liess sich folgern, *dass sich in allen oberflächlichen Erdschichten hierzulande und in Indien Sarcinen vorfinden*, dass aber verschiedene Bodenarten dafür oft ganz verschiedene Säuregrade der Anreicherungsflüssigkeit brauchen. Während manche Bodenarten angesäuerte Malzwürze verlangen und in ungesäuerter Flüssigkeit nur Bakterien wachsen lassen, ist für andere Arten jede Ansäuerung vom Uebel. Es stellte sich als möglich heraus, für jedes Bodenmuster der Erdoberfläche denjenigen Säuregrad auszusuchen, wobei Sarcinenentwicklung möglich war und also zu dem obigen Ausspruch von der Allgegenwart zu gelangen. Hier und in Indien in grösserer Tiefe genommene Muster stellen sich als sarcinenfrei heraus.

Die von diesen Tatsachen eingegebene Erklärung, als sollte ihre Anwesenheit an der Erdoberfläche mit der Düngung zusammenhängen, konnte nicht richtig sein, da sich einerseits in normalen menschlichen und tierischen Fäkalien nie Sarcinen zeigten, anderseits in gänzlich unkultiviertem Sandboden: Dünensand, sowohl am Strande als in den innern Dünen, diese Mikroben sich immer fanden.

Diese Wahrnehmungen waren die Veranlassung zur Untersuchung von Sandmustern verschiedener Herkunft: Sand aus einer öffentlichen Anlage in Amsterdam, für Strassenbauzwecke herbeigeführter Sand, Heidesand, und Sand aus einem Sandbett der Wasserleitung in Helmond. Negativ waren nur das unter den Plaggen gesammelte Muster Heidesand und einige Muster Flusssand, die sich durch Reinheit auszeichneten, und wobei man also die Abwesenheit einer Infektion mit dem Schmutz der Erdoberfläche voraussetzen muss, ebenso wie dies der Fall ist mit dem Muster Dünensand, an einer Stelle genommen, wo vor kurzem ein Sandrutsch stattgefunden hatte, wodurch auf einer Tiefe von ungefähr 2 M. eine Schicht blossgelegt worden war. Dagegen waren, wie bereits bemerkt wurde, die Muster des bewohnten und unbewohnten Meeresstrandes immer positiv.

Ferner konstatierte ich ihre Anwesenheit bis in tiefen Schichten des Wasserfilters in Helmond, der die letzten 4 Jahre ohne Sanderneuerung im Betrieb gewesen war und bei dem Infektion mit Erde nur möglich gewesen ist durch das Wasser selbst, das vor der Filtrierung ein langes, offenes Dekantationsbassin durchläuft. Es sieht danach aus, dass Wind und Staub bei der Verbreitung wohl einen bedeutenden Anteil haben. Dies müsste also ihre allgemeine Anwesenheit in der Luft mit sich bringen, was bisher nicht konstatiert worden war. Von vornherein stand das Vorkommen der schweren, grosszelligen Sarcinen in der Atmosphäre auch nicht zu erwarten. Dennoch liess sich ihre Anwesenheit darin aufzeigen, wozu ich eine Schüssel mit sterilem Sand auf einen offenen, vom Laboratorium durch eine Glastür getrennten Balkon stellte. Nachdem dieser dort ungefähr 3 Wochen dem Wind und Wetter ausgesetzt gewesen war, zeigte es sich, dass der erst negativ reagierende Sand positiv geworden war.

Zweifellos spielen Wind und Staub gleichfalls eine grosse Rolle bei der Verbreitung von *S. maxima* auf Getreidearten, wo sie sich so häufig

zeigen. Um sich davon zu überzeugen braucht man nur verschiedene Kleie-arten, wie früher beschrieben wurde [1]), mit angesäuerter Saccharoselösung bei 37° zu kultivieren. Besonders günstig erwies sich Roggenkleie.

Die Tatsache, dass dies stets bis auf 1 Mal, *S. maxima* war, während aus allen Mustern Erde (2 ausgenommen), Schlamm, Abwasser, Sand und auch aus dem menschlichen Magen *S. ventriculi* isoliert wurde, ist zu auffallend, um nicht einen Augenblick dabei stillzustehen. Man könnte meinen, dass die verschiedenen Kulturflüssigkeiten (angesäuerte Malzwürze für Erde u.s.w., angesäuerte Saccharoselösung für Kleie) für das verschiedene Resultat verantwortlich sein könnten. Die Möglichkeit lässt sich indertat nicht leugnen, und folgende Wahrnehmung spricht auch für diese Auffassung.

Während ja aus einem bestimmten Kleiemuster, bei Kultivierung in der beschriebenen Weise in angesäuerter Saccharoselösung, nur einmal aus vielen *S. ventriculi* isoliert wurde, gewann ich diese Art dreimal hinter einander aus der gleichen Kleie, wenn sie in der Weise wie Sand, Erde u.s.w. in Malzwürze ($p_H = 2.2$) gezüchtet wurde. Die Kleie beherbergt also, wie zu erwarten war, beide Arten und die verschiedenen Nahrungsstoffe sind offenbar die Ursache der verschiedenen Ergebnisse. Dass sie dies nicht unter allen Umständen sind, beweisen obengenannte Ausnahmefälle mit Erde, wobei *S. maxima* auch in Malzwürze gewonnen wurde, wobei man noch die Wahrnehmung hinzufügen kann, dass auch aus dem halbverdauten Mageninhalt eines Kaninchens in dieser Weise *S. maxima* gezüchtet werden konnte. Dagegen ist es uns nie gelungen, diese Art aus Erde, Sand u.s.w. mit Saccharoselösung zu gewinnen, auch nicht wenn sterilisierte Kleie in der gebräuchlichen Menge hinzugefügt wurde. *S. maxima* scheint daher ausser auf Getreide-arten nicht stark verbreitet zu sein.

Ueber die Weise, wie *S. ventriculi* in den Magen kommt, kann wenig Zweifel bestehen. Dass sie kein ständiger Bewohner ist, steht fest: in gesunden Magen findet sie sich nicht und sie muss daher als ein Gelegenheitsgast betrachtet werden, mit der Nahrung mitgekommen, der seine Gegenwart nur kundtut, wenn er die Gelegenheit sich zu entwickeln bekommt, wie dies der Fall ist, wenn die Nahrung länger als normal im Magen verharrt und genügend freie Salzsäure anwesend ist. Infolge der oben angezeigten Anwesenheit in der Luft des Dunstkreises lässt sich das Eindringen in den Magen mit der Nahrung sehr gut erklären. Dass man mikroskopisch und kulturell immer *S. ventriculi* findet und nie *S. maxima*, bleibt inzwischen unaufgeklärt.

Die weite Verbreitung über die Erdoberfläche bringt mit sich, dass auch aus Amsterdamer Grabenwasser einige Male *Sarc. ventriculi* gezüchtet werden konnte; mit Abwasser gelang dies stets, ebenso wie mit Abwasserschlamm, „activated sludge", Meerschlamm aus der Südersee und ähnlichem Material. Ich halte es denn auch wohl für wahrscheinlich, dass die nie

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[1]) l.c.

gezüchtete und nur einige Male abgebildete *S. paludosa* (SCHRÖTER) [1]) mit der Gärungssarcine identisch ist. Ich selbst nahm sie einige Male in Abwasser wahr und fand ihr Aeusseres und ihren Umfang indertat vollkommen gleich.

Von noch einer andern Sarcinenart ist die Anwesenheit in Schlamm und Abwasser beschrieben worden und zwar von der Methansarcine, die einige fettsaure Salze in Methan, Kohlensäure und Kohlensäuresalze umwandelt [2]). Ihr Aussehen ist einigermassen dem der Gärungssarcine ähnlich und es ist auffallend, dass man letztere auch stets antrifft in den mehr oder weniger gereinigten Rohkulturen der Methangärung. Diese Sarcinen hat man aber nie in Reinkultur erhalten, was mit der *S. ventriculi* wohl der Fall ist und letztere sind auch nicht zur Methangärung in dazu geeigneten Acetatflüssigkeiten imstande, wie es sich herausstellte. Ebensowenig konnte ich diese Gärung erhalten mit Sandmustern, mit denen man wohl Entwicklung der *S. ventriculi* bewirkt. Die Folgerung ist daher berechtigt, dass die letzte Art in den Methankulturen nur als Verunreinigung auftrete und dass beide Arten ganz gewiss verschieden seien.

Einen ganz andern Blick auf die Anwesenheit und die Verbreitung der Gärungssarcinen in der Natur gewann ich aber, als ich es versuchte, die Anwesenheit auch auf mikroskopischem Wege in den Materialen darzutun, in denen sie den Kulturexperimenten zufolge so häufig vorkommen. Erde und Kleie waren dafür, aus früher (l.c.) genannten Gründen ungeeignet.

Als ich nun die leichte Kultivierbarkeit aus allerlei Sandmustern gefunden hatte, wurden damit ähnliche Versuche in der Hoffnung auf bessern Erfolg wiederholt, besonders als es sich mir zeigte, dass man solche Muster unter einem tüchtigen Wasserstrahl reinwaschen und weiter einige Monate unter Wasser aufbewahren kann, ohne dass dadurch das Vermögen der Sarcinengärung eingebüsst wird. Ich hielt es daher für wahrscheinlich, dass in dem so behandelten, von allem feinen Schmutz befreiten Sande, in dem alle Organismen genügend Gelegenheit bekommen hatten, sich mit Feuchtigkeit zu sättigen, durch genaues Mikroskopieren Sarcinenpakete oder etwa Kokken von einer in diesen Paketen vorkommenden Grösse ($2\frac{1}{2}$ bis 3 μ), aufgezeigt werden müssten.

Diese Erwartung erfüllte sich aber keineswegs. Wurde der rohe Sand mit Wasser geschüttelt und nachdem er sich gesetzt, die obenstehende Flüssigkeit centrifugiert, so war zwar in den Anreicherungsversuchen das Centrifugat stets positiv, aber das mikroskopische Präparat trug ebenso wenig etwas ein als die langwierigen Versuche, in dem reinen, noch positiv reagierenden Sande die gesuchten Organismen zu sehen.

In der Hoffnung, in dieser schwierigen Sache etwas weiter zu schreiten,

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1) SCHRÖTER, Kryptogamenflora Schlesiens.
WILHELMI, Komp. d. Biol. Beurt. d. Wasser.
WEYL's Handbuch d. Hygiene, II. Band, 3. Atb.

2) N. L. SÖHNGEN, Dissertation, Delft 1906.

versuchte ich der Entwicklung der Sarcinen auf dem Fusse zu folgen, um auf diese Weise etwaige junge Entwicklungsstadien beobachten zu können. Während nun ungefähr 16 Stunden nach dem Anfang des Experiments die Gärung bemerkbar wurde und eine grosse Anzahl Sarcinenpakete sich gebildet hatte, konnten schon nach der 11. Stunde die ersten Pakete gefunden werden. Sie haben dann schon die Grösse von 8 oder mehr Kokken, während daneben nie Pakete von geringerem Umfang, und ganz bestimmt kein allgemeines Kokkenstadium, wahrzunehmen waren. Unsre Kenntniss über die Form, in der sie auf Naturmaterialen vorkommen, wird durch diese Versuche nicht erweitert. Man kann bloss mit Bestimmtheit sagen, dass diese Form eine andre und viel widerstandsfähigere sein muss als wir sie in unsern Reinkulturen kennen, denn da haben sie nur eine Lebensdauer von 2 Tagen, während Naturmateriale in unbeschränkter Zeitdauer für den Sarcinenversuch dienlich sind.

Man könnte dabei zunächst an Sporen denken. Die Zahl der sporenbildenden Sarcinenarten ist gering: BEIJERINCK erwähnt die Bildung bei *S. ureae*, während auch *S. pulmonum* (HAUSER) sporentragend zu sein scheint. Bei *S. ventriculi* hat man sie nie wahrgenommen, auch nicht SURINGAR bei seinen äusserst sorgfältigen Untersuchungen. Ich selbst habe sie auch nie finden können, weder in lebendigen noch in gefärbten Präparaten. In unsern Kulturen fehlt den Sarcinen gewiss die Zeit sie zu bilden, denn sie sind schon lange vorher abgestorben. Dass die Gelegenheit in der Natur besser wäre, lässt sich nicht dartun, solange ihr Lebenszyklus noch so ungenügend bekannt ist.

Man darf ruhig annehmen, dass bei Sarcinenpaketen, die in die Aussenwelt geraten, bald ein Absterben stattfindet, ohne Sporenbildung. Die baldige Sterilität der Sarcinenkulturen mit sterilisierter Erde oder sterilisiertem Sand, beweist das. Etwaige Sporen müssten also von der resistenten Form hervorgebracht werden.

Dass die resistente Naturform nicht die Sarcinenform sein kann wird noch durch nachfolgenden merkwürdigen Versuch bestätigt. Lässt man das Wachsen der Sarcinen in Malzwürze in Gegenwart von Sand, Kleie, Kreide oder Erde, die steril sind, stattfinden und befreit man nachher diese Stoffe durch wiederholte Waschungen mit sterilem Wasser von der Kulturflüssigkeit, so verrät das mikroskopische Bild dieser ausgewaschenen Stoffe die Anwesenheit von einer grossen Zahl Sarcinenpaketen. Impft man diese Stoffe in neue Nahrungsflüssigkeit, so lässt sich auch indertat die Sarcine wieder zum Wachstum bringen, und das gleiche ist bisweilen auch am folgenden Tage noch der Fall. Nach 48 Stunden sind aber die Pakete zu einer weitern Entwicklung nicht mehr imstande, während das mikroskopische Bild noch unverändert ist. Trocknet man den Sand nach dem Auswaschen der Nahrungsflüssigkeit, entweder an der Luft oder in einer Kohlensäureatmosphäre und bei niedriger Temperatur, so bleiben die Sarcinenpakete gut sichtbar, aber die Lebensdauer ist nicht länger als in nassem Zustande. Es gelingt nicht, durch Reiben in einem Mörser die Pakete im reinge-

waschenen Sande zu Kokken zu reduzieren. Wird der trockne Sand stark gerieben so werden sie wohl in Fetzen zerrissen, fallen aber nicht in Kokken aus einander. Verlängerung der Lebensdauer wird durch alles dies nicht erzielt. Dagegen ergiebt das Trocknen eines Musters ausgewaschenen Fluss- oder Meersandes ein Präparat, in dem sich kein einziges Paket entdecken lässt, das aber trotzdem, und sogar Jahre nachher, in angesäuerter Malzwürze eine reiche Sarcinenkultur veranlasst.

Unternimmt man den gleichen Versuch mit nicht-sterilem, sarcinenhaltigem Sand, so ist das Ergebnis das gleiche: sind einmal die Sarcinen zum Wachsen gebracht, so ist ihre Lebensdauer nicht länger als 2 Tage.

Infolge dieser starken Empfindlichkeit der Sarcinenkulturen gegen das Aufbewahren, die es nötig macht, die Reinkulturen alle 2 Tage zu übertragen, darf man, glaube ich, die Folgerung ziehen, *dass die Sarcinenpakete eine sehr empfindliche Form dieses Bakteriums darstellen, deren Abwesenheit man in denjenigen Naturmaterialen ruhig annehmen darf, die auch nach langem Aufbewahren noch positiv auf die Sarcinenprobe reagieren. Darin kommt sie offenbar in einer latenten und sehr widerstandsfähigen Form vor, deren mikroskopische Gestalt noch nicht festgestellt werden konnte. Lässt man sie sich zu Sarcinenpaketen auswachsen, so ist damit die Haltbarkeit auch ganz verschwunden.* Es gelingt auch nicht, die einmal gebildeten Pakete wieder zur haltbaren Form zurückzubringen: fügt man den lebenden Kulturen Erde, Sand, Kreide oder Kleie, die steril sein müssen, bei, so verlängert man die Lebensdauer nur unbedeutend. Nur wenn man die Röhren auf dem Höhepunkt der Gärung mit Kohlensäuregas anfüllt und sie darauf zuschmilzt, kann man die Lebensdauer auf wenige Wochen ausdehnen. Aber auch dieses Mittel ist nicht unfehlbar.

Es kam mir nun wichtig vor zu beobachten, ob sich auch in andrer Hinsicht ein Resistenzunterschied zwischen den Sarcinenpaketen und der latenten Form zeigte. Dazu wurde das Verhalten bei Erhitzung in beiden Fällen beobachtet. Die Reinkulturen der Paketform, in Malzwürze mit Kreide gewachsen, stellten sich als sehr empfindlich heraus: schon 30 Min. bei 50° und 10 Min. bei 55° C. war tödtlich. Der rohe Sand dagegen musste während 10 Min. auf 75° C. erhitzt werden, bevor er das Vermögen verlor, in saurer Malzwürze Sarcinenwachstum zu ergeben. Die Versuche wurden so ausgeführt, dass in Reagenzgläsern ungefähr 3 cc Sand mit 10 cc Wasser zusammengefügt wurden. Diese Reagenzgläser wurden dann in einem Wasserbad von der bestimmten Temperatur getaucht. Kontrollbeobachtungen ergaben, dass unter diesen Umständen innerhalb 3 Min., bei Schütteln in noch kürzerer Zeit, Wasser und Sand die Temperatur des Wasserbades angenommen hatten. Die angegebene Zeitdauer ist die des jeweiligen Untertauchens.

Um aber noch etwaige Unterschiede in der Resistenz durch die Anwesenheit des Sandes zu beseitigen, wurden die Versuche in etwas andrer Weise wiederholt. Einerseits wurde roher Sand mit Wasser geschüttelt

und, nachdem er sich gesetzt, das Oberste abgegossen. Die sehr trübe Flüssigkeit wurde in Reagenzgläser gefüllt und ergab darin einen Niederschlag, dessen Volumen und Feinheit denen der Kreide in den Röhren der Reinkulturen zu vergleichen waren. Eine andre Reihe von Röhren wurde folgendermassen behandelt. Nachdem sich die Sarcinen in Röhren mit Malzwürze, Kreide und sterilem Sand entwickelt hatten, wurde die Flüssigkeit abgegossen und der Sand durch wiederholtes Waschen mit sterilem Wasser gereinigt. Er enthielt dann noch eine grosse Anzahl von Paketen. Es wurden nun wieder 10 cc Wasser in die Röhren geschenkt und die Behandlung fand weiter in der gleichen Weise statt wie bei den Versuchen mit rohem Sand. Ebenso wie dort, wurde das Wasser nach Erhitzung und geschwinder Abkühlung abgegossen und durch Malzwürze ersetzt.

Untenstehende Tabelle giebt eine Uebersicht von den Ergebnissen dieser 4 Versuchsreihen.

Resistenz von S. ventriculi gegen Erhitzung.

Zeitdauer	Temper.	Anzahl Röhren positiv			
		Reinkultur in Malzwürze und Kreide (Sarcinenform)	Satz von rohem Sand (latente Form)	Reinkultur in sterilem Sand (Sarcinenform)	Roher Sand (latente Form)
15 min.	50°	5 von 6	—	—	—
15 „	55°	keine	—	2 von 5	—
30 „	55°	keine	—	1 von 6	—
10 „	60°	keine	6 von 6	2 von 5	—
20 „	60°	keine	6 von 6	1 von 6	6 von 6
10 „	65°	keine	6 von 6	keine	6 von 6
10 „	70°	keine	6 von 6	keine	2 von 6
10 „	75°	keine	4 von 6	keine	keine
10 „	80°	keine	keine	keine	keine

Es erhellt daraus, dass die latente Form, sowohl im rohen Sande als im sandfreien Besatz, offensichtlich widerstandsfähiger ist als die Sarcinenform unter den gleichen Umständen.

Ferner musste noch die Frage beantwortet werden, ob vielleicht im Sande Organismen vorkommen, die, wiewohl sie selbst keine Sarcinenform besitzen, diese dennoch bei ihrer Entwicklung in saurer Malzwürze hervorbringen können. Dabei muss man den etwa aufsteigenden Gedanken, als hätte man hier mit Kunstprodukten zu schaffen, die in den stark sauren Flüssigkeiten entstünden, fahren lassen, wenn man bedenkt, dass auch aus Kleie in neutralem Zuckerwasser gute Kulturen von *S. maxima* zu erhalten sind, während auch die Reinkulturen in neutralen Flüssigkeiten

vollkommen normal wachsen und gären. Wohl wäre es möglich, dass die Pakete die anaerobe Form aerober Organismen andrer Gestalt darstellten. Dass dafür nur die Kokkenform oder die Sarcinenform in Betracht kommt, liegt auf der Hand. Die Frage tat tich also auf: lassen sich in Sand oder sonstwo Kokken oder Sarcinen finden, die unter anaeroben Verhältnissen in Malzwürze das Bild und die Gasbilding der Gärungssarcinen zeigen?

Ich säte nun eine Menge gewaschenen Sand, bei der sich die Möglichkeit der Sarcinengärung herausgestellt hatte, in Platten und in hohen Säulen Malzagar aus. Unter der grossen Zahl von Kolonien, die ich erhielt, kamen viele Kokken vor, die isoliert und dann auf ihr Gärungsvermögen hin untersucht wurden. Daneben wurden alle in diesem Laboratorium anwesenden Arten von Mikrokokken und Sarcinen und noch einige Arten dieser letztern, die erhalten worden waren auf Agarplatten, welche der Luftinfektion ausgesetzt gewesen waren, in die Untersuchung hineinbezogen. Das Ergebnis war vollständig negativ. Mit einigem Staunen stellte ich die Tatsache fest, dass sich in keiner der Kulturplatten und in keiner der Agarröhren Kolonien der Gärungssarcinen fanden, wiewohl doch die Möglichkeit der Koloniebildung unter beiden Umständen feststeht, wofern man von gut gärenden Flüssigkeitskulturen ausgeht. Koloniebildung aus der latenten Form findet also offenbar viel schwieriger statt, obgleich man bei der Beurteilung dieser Frage zu bedenken hat, dass auch bei der Aussaat von gut gärendem Material ein sehr grosser Prozentsatz unverändert in den Platten liegen bleibt und nur ein sehr geringer Teil sich zu Kolonien auswächst.

Hat vielleicht die unsichtbare, latente Form der Sarcine die Abmessungen eines filtrierbaren Organismus? Wie unwahrscheinlich dies auch erscheinen mag, gerade bei diesem grosszelligen Organismus, es war dennoch notwendig, auch darüber Gewissheit zu erlangen.

Nun stellte es sich heraus, dass sich die Organismen von Material wie Sand nicht leicht loslösen: sogar wenn man den Sand mit Wasser in einem Mörser reibt, das obere schmutzige Wasser abgiesst und sich setzen lässt, so sind der ausgewaschene Sand und auch der Satz positiv, aber mit der klaren Flüssigkeit ist keine Sarcinengärung zu erzielen. Filtriert man dann auch das genannte schmutzige Wasser durch eine sterile Kerze, so bleibt mit diesem Filtrat die Gärung ebenfalls aus. So ist daher obenstehende Frage zu verneinen.

Zum Schlusse: ist es möglich, die Bedingungen zu finden, wobei sich in Stoffen wie Sand Zunahme der „latenten" Sarcinen zeigt? Diese Frage wurde von der Erwägung eingegeben, dass ein sporenfreier Organismus, er mag so widerstandsfähig sein wie er will, doch nicht so allgemein in der Natur vorkommen kann und dort unter so stark wechselnden Umständen sein Dasein zu jeder Zeit behaupten kann, ohne dass Vermehrung stattfindet, wodurch sich der Vorrat wieder anfüllt. Zweifellos wirkt der Wind mit bei der Verbreitung dieser Bakterie, doch auch dies

erklärt ihre unbedingte Allgegenwart ungenügend, wenn man nicht auch auf ihre Vermehrung rechnen darf.

Sich auswachsen zu Sarcinen, wofür Zucker und „Pepton" nötig ist, kann hierbei keine Rolle spielen, da dies die Resistenz praktisch auf Null herabsetzt. Eine Vermehrung der Bakterie kann also nur dann eine bleibende erhöhte Anzahl bewirken, wenn auch die Nachkommenschaft die resistente Form besitzt. Vermehrung derselben wird sich zeigen, indem man zum Sarcinenversuch geeigneten Sand unter verschiedenen mehr oder weniger naturähnlichen Umständen aufbewahrt und jedesmal die Menge Sand bestimmt, womit ein positives Ergebnis dieses Versuches zu erhalten ist. Nähme diese Menge merklich ab, so wäre damit eine Vermehrung der latenten Keime bewiesen.

Versuche diese Vermehrung aufzuzeigen, wurden folgendermassen gemacht. Eine grosse Menge Sand wurde durch Waschen vom schlimmsten Schmutz befreit, nachher bei niedriger Temperatur getrocknet und gut gemischt. Von diesem Sand wurden Mengen von 50 gr. in sterilen Kolben mit 50 cc von einer der Lösungen, die man in untenstehender Tabelle findet, übergossen. Nach einer Woche, bei Zimmertemperatur, wurde dann die Flüssigkeit abgesogen und der Sand mit wenig Wasser nachgewaschen und wieder bei niedriger Temperatur getrocknet. Von den so erhaltenen Sandmustern wurden dann je 6 Portionen von 3 gr. und 6 Portionen von 2 gr. in sterilen Stöpselflaschen von 50 cc abgewogen und diese wurden weiter mit Malzwürze angefüllt, die mit Salzsäure angesäuert war zu $p_H = 1.75$. Die Tabelle giebt die Ergebnisse dieser Versuche, wobei noch zu

Art der Flüssigkeit	6 × 3 gr. Sand Anzahl posit.	6 × 2 gr. Sand Anzahl posit.	Mittl. Sarc. titer	Mikr. Bild nach Behandlung	Bemerkungen
1. Ursprünglicher Sand, gewaschen und getrocknet	3	1	7.5 gr.	keine Sarc.	
2. Leitungswasser	4	3	4.3 „	idem	Wasser blieb klar
3. idem + 0.01 % Pepton	3	0	10 „	„	sehr geringe Trübung
4. idem + 0.1 % Pepton	5	5	3 „	„	schwache Trübung
5. idem + 0.5 % Pepton	4	3	4.3 „	„	starkes Anwachsen der Bakterien (Faulung)
6. idem + 1 % Saccharose	3	1	7.5 „	„	Buttersäure gärung
7. idem + 0.1 % NH_4Cl	3	4	4.3 „	„	Flüssigkeit klar
8. idem + 0.1 % Asparagine	4	3	4.3 „	„	Flüssigkeit klar

erwähnen ist, dass der rohe, nicht ausgewaschene Sand schon mit 300—500 mg die Sarcine zum Wachsen brachte.

Aus diesen Zahlen lässt sich, dünkt mich, folgern, dass unter den gegebenen Verhältnissen von einer starken Vermehrung der latenten Sarcinen keine Rede ist. Es lässt sich nicht leugnen, dass das Aufbewahren unter einer 0.1 Proz. Peptonlösung das Ergebnis der Experimente verbessert (0.5 Proz. Pepton veranlasst ein zu starkes Wachstum der Bakterien, das eine Zunahme der latenten Sarcinen verhindert), aber auch durch Leitungswasser allein geht das Titer schon von 7.5 gr. auf 4.3 gr. rückwärts. Dass durch diese Behandlung keine Sarcinenpakete noch auch irgend welches Vorstadium derselben im Sand sichtbar werden, kann nicht wundernehmen, wenn man an die verhältnismässig grossen Mengen Sand denkt, die zur Sarcinenentwicklung notwendig sind.

Ueber die Lebensweise der Sarcinen in der Natur bringen diese Versuche nur wenig Aufklärung und die vielen Rätsel, welche die geheimnisvolle latente Form umgeben, bleiben gleichfalls ungelöst.

Laboratorium der Hygiene der Universität Amsterdam.

Amsterdam, im März 1928.

Physics. — *On* OSEEN's *theory for the approximate determination of the flow of a fluid with very small friction along a body.* By J. M. BURGERS. (Mededeeling N°. 9 uit het Laboratorium voor Aerodynamica en Hydrodynamica der Technische Hoogeschool te Delft.) (Communicated by Prof. P. EHRENFEST.)

(Communicated at the meeting of December 17, 1927).

§ 1. The determination of the flow of a viscous fluid along a body of given contour is one of the most important problems of hydrodynamics. When we consider the stationary flow along a body at rest and start from the equations of motion for an incompressible viscous fluid, the solution is required of a system of four partial differential equations with the velocity components and the pressure as variables. This solution has to satisfy the boundary conditions, which for the case are that the velocity of the fluid along the surface of the body is zero, and that at infinity the flow asymptotically approaches to a parallel motion with given constant velocity. In the following we will suppose that this latter velocity has the value V and is directed along the negative x-axis. Until now the rigorous solution of these equations presents unsurpassable difficulties. However, many applications of hydrodynamics do not require a solution of general validity, but a solution for the case in which the internal friction of the fluid is very small and may be put nearly equal to zero. We shall confine ourselves to this case.

The so called irrotational flow of DIRICHLET may be considered as the first attempt to find a solution of the problem before us. The deduction of this particular solution is based upon the principle that, when the internal friction of the fluid becomes infinitely small, no objections arise against the supposition that the fluid layers will glide over each other in some places. In consequence, even when the condition that the fluid in contact with the body must be at rest, is observed rigorously, we may accept a tangential velocity differing appreciably from zero at a small distance from the surface. The region in which this velocity appears is separated from the surface by a thin layer in which gliding occurs. Calculating the magnitude of the rotation of the flow in this layer by means of the well known formulae, we shall get a very high value; in other words a strong vortex motion is present in this layer. The thickness of this layer may be supposed the smaller as the friction is less.

The normal velocity component, in contrary to the tangential one, will

not acquire an appreciable value at a very small distance from the surface, as this would be in contradiction with the equation of continuity. Therefore the boundary conditions may be simplified into the single condition that only the normal velocity component has to be zero.

We know that in a fluid without friction vortex motion once present cannot be destroyed, and that at the other hand in the interior of the fluid no vortex motion can arise. The flow being free from rotation at some distance in front of the body, we are led to suppose that it nowhere possesses vortex motion. In this case the flow can be characterized by a potential function. This potential, which we will call Φ_0, is deduced from LAPLACE's equation (which in itself is a consequence of the equation of continuity):

$$\triangle \Phi_0 = 0,$$

and from the boundary conditions:

$$\frac{\partial \Phi_0}{\partial n} = 0 \text{ along the surface of the body;}$$

$$\frac{\partial \Phi_0}{\partial x} = -V, \quad \frac{\partial \Phi_0}{\partial y} = \frac{\partial \Phi_0}{\partial z} = 0 \text{ at infinity.}$$

As is known, these equations admit a solution. Moreover it can be proved easily that a pressure field may be calculated from the velocity distribution in such a way that the general equations of motion are satisfied. The flow determined in this way will be called the DIRICHLET flow.

The experiments proved, however, that this DIRICHLET flow differs appreciably from the real state of motion unless we consider a very thin body; in particular the region of vortex motion occurring in the "wake" down stream is lacking in the theoretical solution, while the pressure distribution calculated from it gives zero resultant, which is in contradiction with the observed facts.

We thus have to seek other approximate solutions in which it is not supposed a priori that the entire flow is free from vortex motion. An example of such a solution is the well known discontinuous motion investigated by HELMHOLTZ, KIRCHHOFF, RAYLEIGH and others. We will not enter into a discussion of this solution, which is exposed in numerous text books and communications, and may be regarded as generally known [1]). On the contrary I would treat in the following lines the theory developed by OSEEN. OSEEN originally started from researches concerning the flow with great friction. In the last years, however, several communications

[1]) Compare f.i. H. LAMB, Hydrodynamics (Cambridge), art. 76, 77, 78; PH. FRANK u. R. VON MISES, Die Differential- und Integralgleichungen der Mechanik und Physik (Braunschweig 1927) II, p. 775 and seq.

both from OSEEN himself and from ZEILON have appeared, treating the case of a very small viscosity [2]).

Although OSEEN has given his theory for three dimensional as well as for two dimensional flow, and ZEILON has developed examples of both cases, I will confine myself in the following lines entirely to the two dimensional case, thereby aiming at the most simple formulation in order to exhibit the striking points as clear as possible.

§ 2. In order to understand the leading principles of the theory we must pay attention to the properties of vortex motion. From the hydrodynamical equations it follows that the vortices are transported by the fluid elements ; the directions of the rotation vectors follow the changes in the orientation of the fluid particles, and the intensity of the rotation is modified in such a way that its product with the cross section of a fluid element retains the same value. From this follows that the strength of a vortex tube is not affected by the flow of the fluid. Moreover the action of the viscosity has to be taken into account, producing a diffusion of the vortex motion, which is gradually spread out over the entire field. The equations of motion show that no vortex motion can arise in the interior of the field as long as the motion remains regular ; the only source of vortex motion has to be looked for in the forces which the surface of a solid body exerts upon the adjacent fluid particles.

We now confine ourselves to the two dimensional motion. Then all vortex vectors are perpendicular to the direction of the flow and we only have to discriminate between positive and negative vorticity ; hence the property just mentioned is simplified into the following one : the vorticity is dragged along with the flow, does not alter its strength, but diffuses at the same time from the fluid particles to which it was bound to the surrounding ones. From this we deduce that in stationary two dimensional flow of a fluid with infinitely small friction (in which case the diffusion of the vorticity becomes infinitely slow), the vorticity will be distributed in such a way that the vortex strength has a constant value along every stream line.

In applying this conclusion to the flow in the vicinity of a body, we can add that the vorticity may be found along those stream lines or parts of stream lines only, which extend from the body down stream : for up stream of the body the flow does not supply any vorticity and the diffusion is so weak that it cannot produce an appreciable extension of the vorticity

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[2]) The most important of these papers are : C. W. OSEEN, Zur Theorie des Flüssigkeitswiderstandes, Nova Acta Reg. Soc. Scient. Upsaliensis (IV. 4) 1914 ; Beiträge zur Hydrodynamik I, Ann. d. Physik **46**, p. 231, 1915 ; and in particular : Hydrodynamik (Bd. I der Sammlung Mathematik in Monographien und Lehrbüchern, Leipzig 1927), p. 211 and seq. ; N. ZEILON, On potential problems in the theory of fluid resistance, Kungl. Svenska Vetenskapsakademiens Handlingar, III Ser. I : 1, 1924 ; Beiträge zur Theorie des asymptotischen Flüssigkeitswiderstandes, Nova Acta Reg. Soc. Scient. Upsaliensis 1927.

against the direction of the flow. Stream lines passing the body at some distance will not carry any vorticity.

This idea seems to render little use when we do not know the course of the stream lines, which are found only when the problem is solved.

Now OSEEN has put the question whether we may get a satisfactory approximation by determining the distribution of the vorticity in the field, starting from a simple flow pattern, which needs not correspond to the real flow. Accepting such an imaginary "transport flow for the vorticity", the lines along which the vorticity is constant could be determined. Then we can try to choose the vorticity in such a way that the field belonging to this vortex distribution, superposed upon a suitable irrotational flow, leads to a flow satisfying the boundary conditions along the surface of the body and the condition:

$$u = -V, \quad v = 0 \text{ at infinity.}$$

We will not investigate this idea in its most general aspect, but will consider a simple case, the first one treated by OSEEN, to which, until now, most attention has been paid.

In this case for the "transport flow" the most simple one imaginable is accepted; viz. a flow with a constant velocity V parallel to the x-axis. This flow enters the body at one side and leaves it at the other. We will consider this as unimportant, as this flow does not represent the solution we look for, but is only a means for arriving at the distribution of the vorticity.

The vortex distribution now becomes very simple. At the upstream side of the body we can only have a thin vortex layer, as the vorticity will not diffuse upstream; in the wake of the body, however, in a region of equal breadth, extending down stream to infinity, vorticity will be present, with a strength independent of x, and therefore being a function of y only (compare fig. 1). We shall write for this function: $\zeta(y)$.

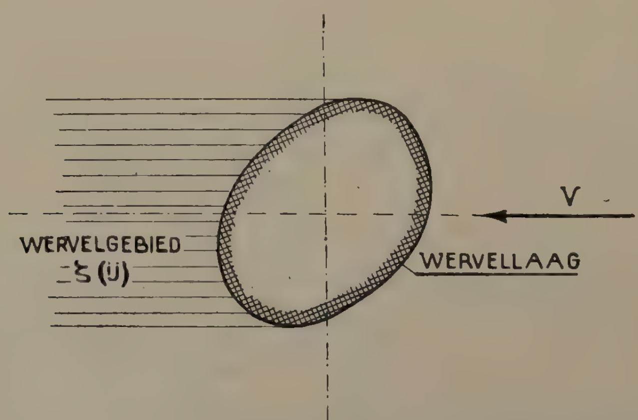

Fig. 1.

When it is preferred to leave aside the idea of a special transport flow for the vorticity, one might as well start from the supposition that in the wake of the body, in a region of equal breadth, the strength of vorticity may be represented by a function $\zeta(y)$, and that a concentrated vortex layer is present at the front side of the body, while outside of these two regions no vorticity occurs in the flow.

Outside of the vortex region we must now get irrotational motion characterized by a potential function $\Phi(x, y)$. We shall suppose that this potential may be continued analytically without singularities into the vortex region in the wake of the body. Then in this region a flow has to be superposed on it determined by the distribution of the vorticity which is zero outside of this region. Writing now in the whole field for the components u and v of the velocity of the resultant flow:

$$u=\frac{\partial \Phi}{\partial x}+u', \quad v=\frac{\partial \Phi}{\partial y}+v' \quad \ldots \ldots \ldots \quad (1)$$

the quantities u' and v' will be zero everywhere outside of the vortex region. Within this region they satisfy the equations

$$\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}=0, \quad \frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y}=\zeta \quad \ldots \ldots \ldots \quad (2)$$

Now it is of importance to remark that we can find the function Φ without knowing in advance the distribution of the vorticity. The boundary conditions tell that the velocity must be zero on the surface of the body. Now an infinitely thin vortex layer is present on the anterior side. Just outside of this layer the condition for the tangential component of the velocity may be left aside, as it can allways be satisfied later on by giving a suitable strength to the vortex layer. Hence only the condition regarding the normal component has to be observed. In this way for the potential, which satisfies the equation of LAPLACE

$$\triangle \Phi=0 \quad \ldots \ldots \ldots \ldots \quad (3)$$

we get:

$$\frac{\partial \Phi}{\partial n}=0 \text{ at the front side of the body} \quad \ldots \ldots \ldots \quad (4)$$

just as in the case of the DIRICHLET flow.

In the wake of the body we have to determine a velocity distribution u', v', that satisfies (2) and is zero everywhere outside of the vortex region. Such a velocity field can be found by supposing that v' is zero everywhere, and that u' is a function of y only, determined by the equation

$$\frac{\partial u'}{\partial y}=-\zeta(y) \quad \ldots \ldots \ldots \ldots \quad (5)$$

When we suppose that over the breadth of the vortex region $\int \zeta \, dy$ has the value zero, equation (5) possesses a solution for u', which is zero at both sides of this region. The fact that ζ does not depend on x of course is of predominant importance; if this was not the case, also u' would depend on x and the equation of continuity would show that v' cannot be put equal to zero.

Both velocity components must vanish at the back of the body. Substituting $v'=0$ in form. (1), we get at the back:

$$\frac{\partial \Phi}{\partial x}+u'=0, \quad \frac{\partial \Phi}{\partial y}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The condition

$$\frac{\partial \Phi}{\partial y}=0 \quad \text{at the back of the body,} \quad . \quad . \quad . \quad . \quad . \quad (6a)$$

together with the condition (4) just mentioned and the conditions:

$$\frac{\partial \Phi}{\partial x}=-V, \quad \frac{\partial \Phi}{\partial y}=0 \quad \text{at infinity} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

are sufficient to determine the potential function Φ; once Φ being found, the relation

$$u'=-\left(\frac{\partial \Phi}{\partial x}\right) \quad \text{at the back of the body} \quad . \quad . \quad . \quad . \quad (6b)$$

gives the velocity component u', from which the vortex field ζ can be deduced.

In this way we come to a new type of potential fields, differing from those of the DIRICHLET flow.

§ 3. As a first example we shall consider the flow around a circular cylinder with radius a. In order to obtain a solution by means of the methods of theory of functions, the complex stream function χ is introduced, which is composed of the potential function Φ and the stream function Ψ:

$$\chi=\Phi+i\,\Psi.$$

Writing $z=x+iy$, we shall get:

$$w=\frac{d\chi}{dz}=\frac{\partial \Phi}{\partial x}+i\frac{\partial \Psi}{\partial x}=u-i\,v=c\,e^{-i\theta},$$

where c represents the absolute value of the velocity, and θ the angle between the direction of flow and the positive x-axis. Putting

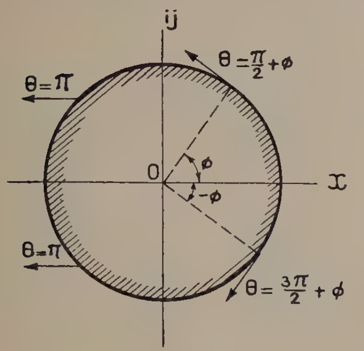

Fig. 2.

$$\omega = lg\, w = lg\, c - i\,\theta. \quad . \quad . \quad (8)$$

ω will be a function for which the solution of the boundary condition problem is much easier than it is for Φ[3]), as the value of the imaginary part of ω, i.e. $-i\,\theta$, is known in all points of the circumference of the circle. In fact the boundary conditions (4) and (6*a*) lead to the expressions (compare fig. 2):

$$\left.\begin{array}{lll} \text{for} & 0 < \phi < \frac{\pi}{2}: & \theta = \frac{\pi}{2} + \phi \\ & \frac{\pi}{2} < \phi < \frac{3\pi}{2}: & \theta = \pi \\ & -\frac{\pi}{2} < \phi < 0: & \theta = \frac{3\pi}{2} + \phi \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Hence the function ω can be found from the integral

$$\omega = -\frac{1}{\pi}\int d\phi\, \frac{d\theta}{d\phi}\, lg\,(z - a\, e^{i\varphi}) + \text{const.} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

taken over the circumference. In our case the particularity occurs that in the point $\phi = 0$ the angle θ changes discontinuously; therefore a term taking into account the amount of the change must be added to the integral.

On account of the relation:

$$\frac{d\theta}{d\phi} = 1 \quad \text{for} \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2},$$

which follows from the data mentioned above, we get

$$\omega = -\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\phi\, lg\,(z - a\, e^{i\varphi}) + lg\,(z - a) + C_1 + i\, C_2 \quad . \quad . \quad (11)$$

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[3]) In treating this example, and also in the following one (the oblique plate) ZEILON uses a theorem given by HILBERT in order to solve the boundary problem for Φ. This leads to a rather tedious calculation.

In a paper by the present author published in these Proceedings, Vol. **23**, p. 1082, 1921, the flow around a cylinder has been calculated by means of a FOURIER expansion for the potential. This method does not converge quickly. Limiting the expansion to 4 terms, for the coefficient in the expression (15) for Q was found: 2,36, with 11 terms: 2,30, the rigorous value being $2(\pi - 2) = 2{,}283$. In the paper mentioned a diagram of the streamlines etc. has been given.

In order to check this formula and at the same time to determine the value of the constant C_2, we make the point z approach to a point $a\,e^{i\Phi_0}$ $(0 < \Phi_0 < \pi)$ of the circle (see fig. 3). We have to bear in mind that the argument of $z - a\,e^{i\Phi}$ takes the value $\frac{1}{2}(\Phi + \Phi_0 + \pi)$ at the points $a\,e^{i\Phi}$ for which $\Phi_0 - \pi < \Phi < \Phi_0$, while it takes the value $\frac{1}{2}(\Phi + \Phi_0 - \pi)$ at the points $a\,e^{i\Phi}$ for which $\Phi_0 + \pi > \Phi > \Phi_0$. The argument of $a\,e^{i\Phi_0} - a$ being equal to $\frac{1}{2}(\Phi_0 + \pi)$, we find for the imaginary part of ω:

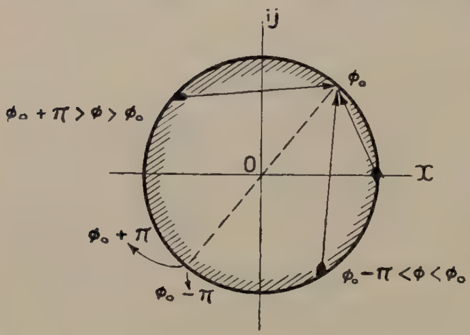

Fig. 3.

$$I(\omega) = -\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\Phi_0} d\Phi\,(\Phi + \Phi_0 + \pi) - \frac{1}{2\pi}\int_{\Phi_0}^{\frac{\pi}{2}} d\Phi\,(\Phi + \Phi_0 - \pi) +$$

$$+ \frac{1}{2}(\Phi_0 + \pi) + C_2 = -\Phi_0 + \frac{\pi}{2} + C_2,$$

when $0 < \Phi_0 < \frac{\pi}{2}$; and:

$$I(\omega) = -\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\Phi\,(\Phi + \Phi_0 + \pi) + \frac{1}{2}(\Phi_0 + \pi) + C_2 = C_2,$$

when $\frac{\pi}{2} < \Phi_0 < \pi$.

Therefore we have to take $C_2 = -\pi$.

At a point on the lower half of the circle, for which $-\pi < \Phi_0 < 0$, the same expression can be used when $\frac{1}{2}(\Phi_0 - \pi)$ is written for the argument of $a\,e^{i\Phi_0} - a$ in stead of $\frac{1}{2}(\Phi_0 + \pi_0)$.

When we calculate the value of ω at a point of the field outside of the circle, the determination of the arguments does not give any difficulty: as soon as the argument of $z - a\,e^{i\Phi}$ has been fixed for any value of Φ, its magnitude for other values, and also the argument of $z - a$ will follow in an unequivocal way.

Considering now the behaviour of the flow at great distances of the cylinder, we may expand the logarithms and shall get:

$$lg\,(z - a\,e^{i\Phi}) = lg\,z - \frac{a\,e^{i\Phi}}{z} - \frac{a^2 e^{2i\Phi}}{2z^2} - \ldots$$

$$lg\,(z - a) = lg\,z - \frac{a}{z} - \frac{a^2}{2z^2} - \ldots$$

which leads to:

$$\omega = -\frac{a}{z}\left(1-\frac{2}{\pi}\right) - \frac{a^2}{2z^2}\dots + C_1 - i\pi.$$

As the velocity at infinity is V, C_1 must have the value $lg\ V$; hence:

$$\omega = lg\ V - i\pi - \frac{\pi-2}{\pi}\frac{a}{z} - \frac{a^2}{2z^2} - \dots \quad \dots \quad (12)$$

From this we find for the value of w:

$$w = u - iv = e^{\omega} = -V\left\{1 - \frac{\pi-2}{\pi}\frac{a}{z} - \dots\right\} \quad \dots \quad (13)$$

and finally we get for the complex stream function:

$$\chi = -V\left\{z - \frac{\pi-2}{\pi} a\, lg\, z - \dots\right\} \quad \dots \quad (14)$$

In the last expression after the term representing a parallel flow with the constant velocity $-V$, a logarithm appears, representing a flow radially outwards, of the total strength

$$Q = 2(\pi-2)\,a\,V = 2{,}283\,a\,V. \quad \dots \quad (15)$$

We can deduce this also from the stream function Ψ, the expression of which is:

$$\Psi = -V\left\{y - \frac{\pi-2}{\pi} a\, arctg\frac{y}{x} - \dots\right\}$$

showing that the stream line $\Psi = 0$, which upstream of the cylinder is directed along the $+x$-axis, is determined down stream by:

$$y = \frac{\pi-2}{\pi} a\, arctg\frac{y}{x} - \dots\dots$$

For $x \rightarrow -\infty$ this equation approaches asymptotically to:

$$y = \pm(\pi-2)\,a = \pm 1{,}14\,a.$$

Both branches lie outside of the vortex region (which is included within the lines $y = +a$ and $y = -a$); therefore they also appear in the complete field.

In order to deduce the value of u', we must calculate the value ot ω

for the points of the semi-circle between $\Phi=\frac{\pi}{2}$ and $\Phi=\frac{3\pi}{2}$. As $\frac{\partial \Phi}{\partial y}$ is zero here, we find u' from the equation:

$$u'=-\frac{\partial \Phi}{\partial x}=-\frac{d\chi}{dz}=-e^{\omega}=e^{R(\omega)}.$$

We can confine ourselves to consider the real part of ω only, as the imaginary part has the constant value $-\pi i$.

Now for $z=a e^{i\Phi_0}$ we have:

$$|z-a e^{i\Phi}|=2 a \sin\frac{\Phi_0-\Phi}{2},$$

therefore we find:

$$R(\omega)=-\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\Phi \lg \sin\frac{\Phi_0-\Phi}{2}+\lg \sin\frac{\Phi_0}{2}+\lg V=$$

$$=\lg V+\lg \sin\frac{\Phi_0}{2}+\frac{1}{\pi}\left(\Phi_0-\frac{\pi}{2}\right)\lg \sin\left(\frac{\Phi_0}{2}-\frac{\pi}{4}\right)-$$

$$-\frac{1}{\pi}\left(\Phi_0+\frac{\pi}{2}\right)\lg \sin\left(\frac{\Phi_0}{2}+\frac{\pi}{4}\right)+\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\Phi\,(\Phi-\Phi_0)\cot\frac{\Phi-\Phi_0}{2}.$$

The last term occurring in this expression can be reduced to a function of the form:

$$\varphi(\xi)=\frac{2}{\pi}\int_0^{\xi} d\xi\,\xi\cot\xi,$$

which has been calculated by ZEILON for values of ξ ranging from 0 till $\frac{3\pi}{4}$ [4]).

Then:

$$\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\Phi\,(\Phi-\Phi_0)\cot\frac{\Phi-\Phi_0}{2}=\varphi\left(\frac{\Phi_0}{2}+\frac{\pi}{4}\right)-\varphi\left(\frac{\Phi_0}{2}-\frac{\pi}{4}\right).$$

Let us consider f.i. the points determined by $\Phi_0=\frac{\pi}{2}$ and $\Phi_0=\pi$.

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[4]) Compare N. ZEILON, On potential problems, etc., p. 34, table I.

According to the table given by ZEILON we have:

$$\varphi(0)=0;\quad \varphi\left(\frac{\pi}{4}\right)=0{,}465;\quad \varphi\left(\frac{\pi}{2}\right)=lg\,2=0{,}693\ ^{5)};\quad \varphi\left(\frac{3\pi}{4}\right)=0{,}229.$$

Therefore the point $\Phi_0=\frac{\pi}{2}$ gives:

$$R(\omega)=lg\,V+\frac{1}{2}\,lg\,2,$$

from which

$$u'_{90^\circ}=V\sqrt{2}=1{,}414\,V.$$

At $\Phi_0=\pi$ we get:

$$R(\omega)=lg\,V+0{,}111,$$

and

$$u'_{180^\circ}=1{,}117\,V.$$

From the fact that u' is not equal to zero for $y=a$, while for values of y just higher than a, u' must be 0, it follows that the line $y=a$ (and also the line $y=-a$) represents a vortex layer, along which ζ becomes infinite in such a way that

$$\lim_{\varepsilon=0}\int_{a-\varepsilon}^{a+\varepsilon}\zeta\,dy=u'_{(y=a)}=V\sqrt{2}.$$

We note the relation:

$$Q=\int_{-a}^{+a}u'\,dy \quad \ldots\ldots\ldots \quad (16)$$

where Q is the quantity determined by (15). This is immediately deduced from the consideration that Q denotes the amount which the flow represented by Φ carries from the cylinder outward to infinity; this

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5) That the integral

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}}d\xi\,\xi\,\cot\xi=-\frac{2}{\pi}\int_0^{\frac{\pi}{2}}d\xi\,lg\,\sin\xi$$

has the value $lg\,2$, is easily proved from the relations:

$$\int_0^{\frac{\pi}{2}}d\xi\,lg\,\sin\xi=\int_0^{\frac{\pi}{2}}d\xi\,lg\,\cos\xi=\frac{1}{2}\int_0^{\frac{\pi}{2}}d\xi\,lg\,\frac{\sin 2\xi}{2}=\frac{1}{4}\int_0^{\pi}d\xi\,lg\,\frac{\sin\xi}{2}=$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}d\xi\,lg\,\frac{\sin\xi}{2}=\frac{1}{2}\int_0^{\frac{\pi}{2}}d\xi\,lg\,\sin\xi-\frac{\pi}{4}\,lg\,2.$$

amount is replenished by the flow u' directed towards the cylinder. Mathematically expressed, when ds denotes an element of the circle:

$$Q = \int ds \frac{\partial \Phi}{\partial n} = -\int_{-a}^{+a} dy \left(\frac{\partial \Phi}{\partial x}\right) \text{ (at the back of the cylinder)} = +\int_{-a}^{+a} u'\, dy.$$

§ 4. As a second example we can take the flow along a plate of breadth $2a$, making an angle α with the x-axis (see fig. 4 where AB represents the plate). In connection with the treatment given by ZEILON we shall represent the flow along AB in the z-plane conformally upon

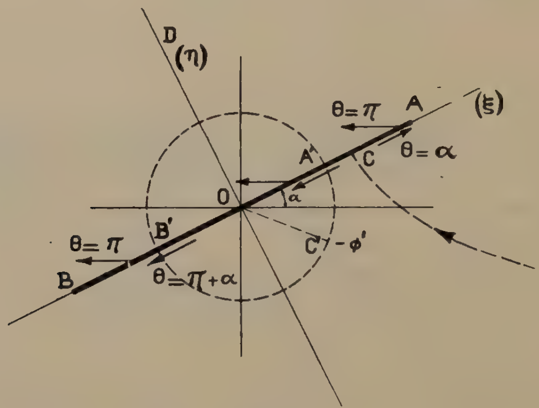

Fig. 4.

the field outside of a circle with radius $\frac{a}{2}$ in a τ-plane, related to the z-plane by means of the formula:

$$z e^{-i\alpha} = \tau + \frac{a^2}{4\tau} \quad \ldots\ldots\ldots \quad (17)$$

In this expression the complex variable $\tau = \xi + i\eta = \frac{a}{2} e^{i\phi}$ is referred to the axes OA and OD. When now we suppose the values which ω takes in the points of both sides of the line AB to be transported to the corresponding points of the circle, we shall get for θ:

$$\left.\begin{array}{ll} \text{along the segment } 0 < \phi < \pi, \text{ corresponding to the back of the plate} & \theta = \pi \\ \text{along the segment corresponding to the front side of the plate:} & \\ \quad \text{for } \pi < \phi < 2\pi - \phi' & \theta = \pi + \alpha \\ \quad \text{for } -\phi' < \phi < 0 & \theta = \alpha \end{array}\right\} \quad (18)$$

where $-\phi'$ is the argument of the point C' corresponding to the stagnation point at the front side of the plate. The position of this point is unknown for the present.

As θ has a constant value in every one of the three intervals defined

above, and only shows abrupt changes at the points A', B, C' respectively of the amounts $\pi-\alpha, \alpha, -\pi$, (10) gives us the three terms only:

$$\omega = -\frac{\pi-\alpha}{\pi}\, lg\left(\tau-\frac{a}{2}\right)-\frac{\alpha}{\pi}\, lg\left(\tau+\frac{a}{2}\right)+lg\left(\tau-\frac{a}{2}e^{-i\Phi'}\right)+C_1+iC_2\,. \quad (19)$$

Applying this formula to the points of the circle, we find: $iC_2 = -i\pi - \frac{1}{2}\, i(\alpha-\Phi')$. When z becomes infinite, τ approaches asymptotically to $z\,e^{-i\alpha}$, and the expansion of ω becomes:

$$\omega = C_1 - i\left(\pi+\frac{\alpha-\Phi'}{2}\right)+\frac{a}{2z}\,e^{i\alpha}\left(\frac{\pi-2\alpha}{\pi}-e^{-i\Phi'}\right)+\ldots$$

The velocity must approach here to: $u=-V$, $v=0$, hence ω has to take the value $lg\, V - i\pi$, from which follows $C_1 = lg\, V$, $\Phi' = \alpha$. Therefore:

$$\omega = lg\, V - i\pi - \frac{a}{2z}\left(1-\frac{\pi-2\alpha}{\pi}\,e^{i\alpha}\right)+\ldots \quad (20)$$

which leads to:

$$w = e^{\omega} = -V\left\{1-\frac{a}{2z}\left(1-\frac{\pi-2\alpha}{\pi}\,e^{i\alpha}\right)+\ldots\right\} \quad (21)$$

and finally:

$$\chi = -V\left\{z-\frac{a}{2}\left(1-\frac{\pi-2\alpha}{\pi}\,e^{i\alpha}\right)lg\, z+\ldots\right\} \quad (22)$$

In this case too a logarithm occurs in χ, now, however, having a complex coefficient. The flow shows a divergence amounting to:

$$Q = \pi a V\left(1-\frac{\pi-2\alpha}{\pi}\cos\alpha\right) \quad (23a)$$

and a circulation of the magnitude:

$$C = a V(\pi-2\alpha)\sin\alpha \quad (23b)$$

Q and C both become zero in the trivial case $\alpha=0$; moreover C is zero, as might be expected, when $\alpha=\frac{\pi}{2}$ [6]. In this case Q becomes $\pi a V$.

We can deduce the vortex strength of the plate by calculating the value of $R(\omega)$ for $\tau=\frac{a}{2}e^{i\Phi}$, where $0<\Phi<\pi$.

§ 5. Besides the distribution of the velocity a knowledge of the pressure is of importance. In calculating the pressure it becomes evident

6) At small values of α the circulation C becomes equal to $\pi a V\alpha$. As ZEILON remarks, this is half the value given by the theory of KUTTA and JOUKOWSKY.

that the flow does not satisfy the exact equations of motion. In ZEILON's first paper, and also in OSEEN's Hydrodynamik the pressure distribution therefore is deduced from the approximate equations on which the calculations of the flow are based.

If the exact equations of motion for the stationary flow are written in the form:

$$\left.\begin{aligned}\frac{\varrho}{2}\frac{\partial}{\partial x}(u^2+v^2)-\varrho v\zeta &= -\frac{\partial p}{\partial x}+\mu\Delta u\\ \frac{\varrho}{2}\frac{\partial}{\partial y}(u^2+v^2)+\varrho u\zeta &= -\frac{\partial p}{\partial y}+\mu\Delta v,\end{aligned}\right\}$$

where, as before,

$$\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y},$$

the approximate equations, which were the starting point of OSEEN's calculations, will be:

$$\left\{\begin{aligned}\frac{\varrho}{2}\frac{\partial}{\partial x}(u^2+v^2)\qquad\qquad &= -\frac{\partial p}{\partial x}+\mu\Delta u\\ \frac{\varrho}{2}\frac{\partial}{\partial y}(u^2+v^2)-\varrho V\zeta &= -\frac{\partial p}{\partial y}+\mu\Delta v\end{aligned}\right\} \quad . \; . \; . \; . \; (25)$$

The difference between the latter equations and the exact ones can be expressed by saying that in the terms in which the vortex intensity ζ occurs multiplied by the velocity, the components of the actual velocity u, v are replaced by the components of the "transport flow of the vorticity" as introduced into § 2, i.e. by $-V$ and 0 respectively.

If μ goes to zero we can integrate the system (25); using (5) we find:

$$p=const.-\frac{1}{2}\varrho(u^2+v^2)-\varrho Vu' \quad . \; . \; . \; . \; . \; (26)$$

For the constant term we take $\frac{1}{2}\varrho V^2$, in order to make p zero at an infinite distance from the vortex region. Along the front side of the body, where $u'=0$, (26) reduces to the ordinary BERNOULLI formula. At the back side we have $u=v=0$ and therefore:

$$p_b=\frac{1}{2}\varrho V^2-\varrho Vu' \quad . \; . \; . \; . \; . \; . \; . \; . \; (26a)$$

In the case of the cylinder u' lies between $V\sqrt{2}$ at $\Phi=90^\circ$ and 1,117 V at $\Phi=180^\circ$; hence p_b is comprised between $-0{,}914\,\varrho V^2$ and $-0{,}617\,\varrho V^2$.

In order to determine the resistance experienced by the body, we put in formula (26):

$$u=\frac{\partial\Phi}{\partial x}+u', \quad v=\frac{\partial\Phi}{\partial y},$$

which gives:

$$p=\frac{1}{2}\varrho V^2-\frac{1}{2}\varrho\left\{\left(\frac{\partial\Phi}{\partial x}\right)^2+\left(\frac{\partial\Phi}{\partial y}\right)^2\right\}-\varrho\frac{\partial\Phi}{\partial x}u'-\frac{1}{2}\varrho u'^2-\varrho V u'.$$

Bearing in mind that $u'=0$ at the front of the body and that $\partial\Phi/\partial x=-u'$ at the back side, we can write for the value of p at the points of the surface of the body:

$$p=-\frac{1}{2}\varrho\left\{\left(\frac{\partial\Phi}{\partial x}\right)^2+\left(\frac{\partial\Phi}{\partial y}\right)^2\right\}+\frac{1}{2}\varrho(u'-V)^2 \quad . \quad . \quad . \quad (27)$$

We shall consider first the term

$$p_I=-\frac{1}{2}\varrho\left\{\left(\frac{\partial\Phi}{\partial x}\right)^2+\left(\frac{\partial\Phi}{\partial y}\right)^2\right\}.$$

By means of some reductions the resultant of this part of the pressure is found to be [7]):

$$\int p_I\cos(n,x)\,ds=-\varrho QV+\varrho\int_{-a}^{+a}u'^2\,dy.$$

(directed along the negative x-axis). The second part of the expression (27) gives a pressure:

$$p_{II\,f}=\frac{1}{2}\varrho V^2,$$

at the front side of the body, and a pressure:

$$p_{II\,b}=\frac{1}{2}\varrho(u'-V)^2;$$

at the back side. The resultant of these pressures has the magnitude (again taken along the negative x-axis):

$$\int_{-a}^{+a}dy\,(p_{II\,f}-p_{II\,b})=\varrho QV-\frac{1}{2}\varrho\int_{-a}^{+a}u'^2\,dy.$$

The total resistance now becomes:

$$W=+\frac{1}{2}\varrho\int_{-a}^{+a}u'^2\,dy \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

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7) In order to prove this formula, it is necessary to add to $\int p_I\cos(n,x)\,ds$ the quantity

$$\varrho\int\left\{\left(\frac{\partial\Phi}{\partial x}\right)^2dy-\frac{\partial\Phi}{dx}\frac{\partial\Phi}{dy}dx\right\},$$

which is again substracted afterwards. Compare ZEILON, On potential problems, p. 17 and OSEEN, Hydrodynamik, p. 295; the calculation is performed here in a slightly different way, as these authors allways consider a moving body within a fluid at rest.

By numerical integration ZEILON deduced for the case of the cylinder:

$$W = 0{,}657\, \varrho\, V^2 d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (28a)$$

where $d = 2a =$ diameter of the cylinder.

In the case of the oblique plate we get a resistance, and besides a force in the direction of y.

We note that on account of the discontinuity of u and of the presence of the term $-\varrho V u'$ formula (25) for the pressure is discontinuous along the lines $y = +a$ and $y = -a$, which confine the vortex region. Proceeding from $y = a + \varepsilon$ to $y = a - \varepsilon$ (where ε is a small positive quantity) in the case of the cylinder u increases from a value $-u_1$ (depending on x) to $-u_1 + V\sqrt{2}$; u' increases from 0 to $V\sqrt{2}$. Therefore the pressure decreases suddenly by the amount $\varrho V^2 (1 + \sqrt{2} - u_1 \sqrt{2}/V)$. From a physical point of view this result is impossible. Of course this is due to the fact that with the supposed transport flow we have to apply a system of special forces to the field of motion, which at the same time influence the distribution of the pressure.

In his second paper ZEILON tried to escape from this difficulty. For the determination of the field of flow it was essential that the vorticity ζ was a function of y only; the magnitude of the "transport velocity" does not affect this function. Therefore the calculations of §§ 2—4 are not modified when we take any function $-U(y)$ for this velocity in stead of $-V$, provided that it is again directed along the y-axis. Then, however, the second equation of (25) is changed into:

$$\frac{\varrho}{2}\frac{\partial}{\partial y}(u^2 + v^2) - \varrho U \zeta = -\frac{\partial p}{\partial y} + \mu \triangle v \quad . \quad . \quad . \quad . \quad . \qquad (29)$$

and for the pressure distribution an other solution is obtained. ZEILON requires that in the immediate vicinity of the body the hydrodynamical equations have to be satisfied as well as possible; hence immediately behind the body the pressure may not change discontinuously when we enter into the vortex region. As the velocity is zero everywhere along the back of the body, and the terms $\mu\triangle u$, $\mu\triangle v$, occurring in the equations of motion, vanish here at the same time with μ, p has to be constant along the back side. It takes the value which is given by the formula (valid at the front side):

$$p_f = \frac{1}{2}\varrho V^2 - \frac{1}{2}\varrho (u^2 + v^2)$$

when applied to the limiting points, where the front transfers into the back.

In the case of the cylinder this value is (with $u_{90^\circ} = V\sqrt{2}$, $v_{90^\circ} = 0$):

$$p_a = -\frac{1}{2}\varrho V^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (30)$$

This value is higher than that which is found from (26); therefore ZEILON finds a smaller resistance:

$$W = 0{,}523\, \varrho V^2 d \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (31)$$

In the case of the plate this consideration, however, is not valid. The flow described by (19) gives an infinitely high velocity at the edges of the plate, and therefore an infinite negative pressure, which cannot be continued as in the case of the cylinder.

ZEILON tried finally to obtain still other pressure distributions by supposing a modified vortex distribution, in which the vortex layer, occurring at the front side of the body extends over some distance on both sides along the back [8]). Along these parts of the back side we no more have to satisfy the boundary conditions: $u=0$, $v=0$, and therefore we cannot decide whether $\partial\Phi/\partial y=0$. ZEILON now introduces a certain hypothesis about the angle θ considered in § 3; then it is still possible to solve the boundary problem in a simple way. By means of an appropriate assumption about the course of θ, ZEILON in the case of the cylinder succeeds in obtaining a pressure distribution which fits the experimental results rather well. However, this modification of the original hypothesis takes away its former simplicity, and contains an element of arbitrariness.

§ 6. We might ask if the resistance could be found by applying the theorem of momentum. As the flow does not satisfy the hydrodynamical equations of motion, and therefore neither the fundamental equations of mechanics, this seems to be impossible.

However, there is a characteristic particularity in the solutions given by OSEEN's theory, which makes the application of this theorem to be not entirely without prospects. Therefore we direct our attention to the infinite part of the field.

According to OSEEN's theory the vorticity is zero everywhere in the field, with the exception of the region extending down stream of the body. Outside of this latter region the flow is characterized by a potential and therefore it also possesses a complex stream function. The latter can be expanded in the following progression:

$$\chi = -Vz + (A_1 - iA_2)\, lg\, z + \frac{B_1 + iB_2}{z} + \quad . \ . \ . \qquad (32)$$

which leads to the potential function (with $r=\sqrt{x^2+y^2}$):

$$\Phi = -Vx + A_1\, lg\, r + A_2\, arctg\, \frac{y}{x} + \frac{B_1 x + B_2 y}{r^2} + \ . \ . \qquad (33)$$

The term $A_1\, lg\, r$, representing a flow directed radially outward of the total strength $Q=2\pi A_1$, is characteristic for the theory. This term does not occur in the case of the DIRICHLET flow. On the contrary the term $A_2\, arctg\, \frac{y}{x}$, representing a circulation, is well known from the aerofoil theory of KUTTA and JOUKOWSKY.

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[8]) ZEILON, Beiträge zur Theorie, . . . p. 35.

As in reality no resultant radial flow can occur, the flow has to be replenished by a flow directed inward, which evidently can occur only in the wake of the body. This is confirmed by the experiments. They show, however, that the breadth of the vortex region does not remain finite, but gradually increases down stream. The compensational flow is therefore spread over a greater extent, and its velocity decreases in inverse proportion.

Let us now start from the supposition that the flow outside of the vortex region may be characterized by a potential function Φ of the type (33) and that within this region it is allowed to put:

$$u = \frac{\partial \Phi}{\partial x} + u', \qquad v = \frac{\partial \Phi}{\partial y} + v' \quad . \quad . \quad . \quad . \quad . \quad . \quad (34)$$

where u' and v' satisfy the equation of continuity:

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

It is not a priori certain that the potential can allways be expressed by a series of the type (33) and then can be continued analytically throughout the entire vortex region. In the case of the discontinuous flow described by HELMHOLTZ and KIRCHHOFF for the case of a plate, transverse to the x-axis, we should have found an expansion for χ, beginning with the terms:

$$\chi = -Vz + \beta \sqrt{z} + \ldots = -Vre^{i\phi} + \beta \sqrt{r}\, e^{\frac{i\phi}{2}} + \ldots$$

where the argument of $\sqrt{z}$ is determined by the condition $-\pi < \phi < +\pi$.[9])

However, we will adhere, be it by the way of a hypothesis, to formula (33).

Now it is necessary to gather some data concerning the flow in the vortex region. Although the limits of this region will not be defined clearly in general, we may suppose that it is wholly included between two curves

$$y = f_1(x), \qquad y = f_2(x).$$

With great probability we may expect that at a great distance down stream of the body the functions f_1 and f_2 satisfy the conditions:

$$f_1 \ll |x|, \qquad f_2 \ll |x|,$$

so that also the breadth $b = f_1 - f_2$ of the vortex region only slowly increases with x. We arrive at this supposition when bearing in mind that at great distances of the body, where eventual irregular motions are damped out, the spreading out of the vortex region is determined only by the frictional forces. At a great distance from the body the velocity components u and v will approach asymptotically to $-V$, 0,

[9]) In the following terms of this series also a logarithm occurs.

so that here OSEEN's equation for the vortex motion:

$$-\varrho V \frac{\partial \zeta}{\partial x} = \mu \triangle \zeta$$

will hold with continually increasing approximation.

The greater part of the following considerations, however, are valid also when the breadth b does not increase indefinitely but reaches a finite limit.

The velocity u' which occurs in the vortex region, is subjected to the relation:

$$\int_{f_1}^{f_2} u' \, dx = Q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

When b increases without limit, u' must decrease indefinitely, as stated above.

If we bear in mind that $\partial u'/\partial x$ is of the order of u'/r, and that according to the equation of continuity (35) the same must hold for $\partial v'/\partial y$, we deduce that in the vortex region the quantity v' is at most of the order of bu'/r. Therefore $\partial v'/\partial x$ will be of the order of bu'/r^2. We now put the equation for $\partial p/dy$ into the following form:

$$\frac{\partial}{\partial y}\left\{p + \frac{\varrho}{2}\left(\frac{\partial \Phi}{\partial x}\right)^2 + \frac{\varrho}{2}\left(\frac{\partial \Phi}{\partial y}\right)^2\right\} =$$

$$= -\varrho\left[\frac{\partial \Phi}{\partial x}\frac{\partial v'}{\partial x} + u'\left(\frac{\partial^2 \Phi}{\partial x \, \partial y} + \frac{\partial v'}{\partial x}\right) + \frac{\partial \Phi}{\partial y}\frac{\partial v'}{\partial y} + v'\left(\frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial v'}{\partial y}\right)\right] + \mu \triangle v'.$$

Investigating the order of magnitude of the terms occurring at the right hand side of this expression, the highest will be found to be bu'^2/r^2. The variations of the expression between $\{ \}$ on the left hand side, along a line drawn parallel to the y-axis through the vortex region, therefore are at most of the order: $b^2 u'^2/r^2$. As $\int u'^2 \, dy$ over the breadth of the vortex region cannot get an infinite value, not only these variations themselves but also the integral of the variations of the expression $\{ \}$ over the breadth of the vortex region will become zero when r increases indefinitely. We may deduce from these considerations that the pressure in the vortex region at great distances from the body will be given with sufficient accuracy by the approximate formula:

$$p = const. - \frac{1}{2}\varrho\left\{\left(\frac{\partial \Phi}{\partial x}\right)^2 + \left(\frac{\partial \Phi}{\partial y}\right)^2\right\} \quad . \quad . \quad . \quad . \quad . \quad (37)$$

After inserting the value of Φ from (33), we get:

$$p = const. + \varrho V \frac{A_1 x - A_2 y}{r^2} + \text{terms of the order } r^{-2} \ldots \quad (37a)$$

We may now apply the theorem of momentum to the region bounded

by a circle with very great radius R around the origin. Then for the x-component of the resultant of the pressure forces on this circle we find:

$$K = \int_0^{2\pi} p\, R \cos \Phi\, d\Phi = \pi \varrho\, V A_1 = \frac{1}{2} \varrho\, V Q$$

(taken in the negative direction of x). The transport of momentum along the $-x$-axis into the region within this circle is:

$$I = \int_0^{2\pi} \varrho\, u\, (u \cos \Phi + v \sin \Phi)\, R\, d\Phi.$$

Inserting from (34), I becomes:

$$I = \varrho \int_0^{2\pi} \left\{ \left(\frac{\partial \Phi}{\partial x}\right)^2 \cos \Phi + \frac{\partial \Phi}{\partial x} \frac{\partial \Phi}{\partial y} \sin \Phi \right\} R\, d\Phi +$$

$$+ \varrho \int_0^{2\pi} \left(2u' \frac{\partial \Phi}{dx} + u'^2 \right) R \cos \Phi\, d\Phi + \varrho \int_0^{2\pi} \left(v' \frac{\partial \Phi}{\partial x} + u' \frac{\partial \Phi}{\partial y} + u' v' \right) R \sin \Phi\, d\Phi.$$

The first integral has the value $-3\pi \varrho V A_1 = -\frac{3}{2} \varrho V Q$. In the second integral we put $R \cos \Phi\, d\Phi = -dy$; as $\int u'\, dy$ is finite, we may take $\partial \Phi / \partial x = -V$ in this formula and get:

$$\varrho \int_{f_1}^{f_2} (2u' V - u'^2)\, dy = 2 \varrho V Q - \varrho \int_{f_1}^{f_2} u'^2\, dy.$$

The greatest one of the terms, occurring in the expression between ( ) in the third integral, is of the order of bu'^2/R; as also $\sin \Phi$ becomes infinitely small for the points of the vortex region, the integral may be neglected entirely.

The resistance experienced by the body is equal to the sum of K and I. Hence finally we find:

$$W = \varrho V Q - \varrho \int_{f_1}^{f_2} u'^2\, dy \quad \ldots \ldots \quad (38)$$

If we adhere to the supposition that the breadth of the vortex region increases indefinitely down stream, the second term expires; therefore:

$$W = \varrho V Q \text{ [10]} \quad \ldots \ldots \quad (38a)$$

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[10]) When the compensational flow u', v' is not introduced, we find: $K + I = -\rho\, VQ$. This result has been obtained by LAGALLY; compare M. LAGALLY, Zeitschr. f. angew. Mathematik u. Mechanik, **2**, p. 409, 1922.

The appearance of the resistance therefore is essentially connected with the presence of a logarithmic term in the potential. And this term is obtained only then, when $\partial\Phi/\partial n$ is not zero at all points of the surface of the body.

In accordance with this result is the experimental observation that the streamlines which follow the contour of the body at the front side, leave this contour at certain points, usually situated in the neighbourhood of the section of maximum breadth or of a sharp edge. The points where this occurs limit the region in which $\partial\Phi/\partial n = 0$; beyond these points we evidently have $\partial\Phi/\partial n > 0$.

In the case of a cylinder the potential determined from the boundary conditions (4) and (6*a*) of § 2, according to OSEEN's theory, gives $Q = 2{,}283\, aV$, which leads to a much too high value of the resistance. Hence we must look for other boundary conditions for the potential; and the modification of DIRICHLET's condition is to be varied in such a way, that Q decreases. The researches of ZEILON mentioned at the end of § 5, possibly constitute a beginning in this direction.

In deducing form. (38*a*) it is supposed of course that in the infinite region of the field the flow determined by the functions Φ and u' approaches to the flow that exactly satisfies the true equations of motion. However, it is not required that this flow is entirely stationary; the deductions also hold when the flow in the immediate vicinity of the body fluctuates about a certain mean value, if only the fluctuations decrease without limit when we go farther and farther away from the body. The formula then gives the mean value of the resistance.

Mathematics. — *The Congruence of the Twisted Cubics that Cut Five given Lines Twice.* By Prof. JAN DE VRIES.

(Communicated at the meeting of February 25, 1928).

§ 1. The twisted cubics k^3 that have the lines b_1, b_2, b_3, b_4 and b_5 as bisecants, form a congruence Γ. In order to represent this on a field of points we establish a projective correspondence between the point range (C) on b_1 and the tangents (c) of a conic c^2. As the image of the k^3 that cuts b_1 in C_1 and C_2, we shall consider the point of intersection F of the tangents c_1 and c_2 corresponding to C_1 and C_2.

The k^3 that cut a line l, form a system Λ of which we shall determine the image curve.

The k^3 through C_1 that have b_2, b_3, b_4, b_5 as bisecants, form a congruence, which has been investigated by GODEAUX [1]). The k^3 of this congruence that rest on l, form a surface of the 9th degree with a triple point C_1. There are, accordingly, *six* k^3 that cut b_1 in one more point C_2. Consequently the *system* Λ is *represented* in a *curve* λ^6 and the k^3 through C_1 form a surface $(C)^6$.

§ 2. Any conic k^2 that cuts a line b_k twice and rests on the four other lines b, is completed by each of the two transversals t'_k, t''_k of these four b to a k^3 of the congruence Γ.

The curves k^2 in planes through b_1 that cut b_2, b_3, b_4, b_5 and m, form a surface O^8. The line t'_1 cuts O^8 in four points that do not lie on a line b; hence the k^2 that cut b_1 twice and rest on b_2, b_3, b_4, b_5 and t'_1, form a surface O'^4_1 with *double line* b_1.

On b_1 these k^2 define a correspondence (2, 2) between the points C_1 and C_2; the system Σ'_1 of the figures (k^2, t'_1) is, therefore, represented on a conic ω'^2_1. Its points of intersection with c^2 are the images of the figures that touch b_1.

There is no k^2 with bisecant b_1 that rests on the other four b and at the same time on the transversals t'_1 and t''_1. Accordingly any point of intersection of the conics ω'^2_1 and ω''^2_1 is the image of more than one k^3 and is, therefore, a *singular point*.

§ 3. This is confirmed by the following consideration. The *hyperboloid* H_{234} with directrices b_2, b_3, b_4 contains the transversals t'_5 and t''_5 of

[1]) L. GODEAUX, *Sur une congruence linéo-linéaire de cubiques gauches* (Bull. Acad. de Belgique 1908, N^0. 4, p. 531). This congruence can also be investigated by the aid of the above mentioned representation; however this has no singular points.

b_1, b_2, b_3, b_4; their points of intersection with b_1 are indicated by C'_{1234} and C''_{1234}. If B'_{2345} and B''_{2345} are the points of intersection of H_{234} and b_5, H_{234}, contains a system of ∞^1 curves k^3 that pass through C'_{1234}, C''_{1234}, B'_{2345}, and B''_{2345}, cut b_2, b_3, b_4 twice, and belong, therefore, to the congruence Γ[1]). All these k^3 are represented in the *singular point* $S_5 \equiv c'_5\, c''_5$ defined by C'_{1234} and C''_{1234}.

Analogously we find the *singular points* S_2, S_3 and S_4 as images of the systems on the *hyperboloids* H_{345}, H_{245} and H_{235}.

The four points S_k are evidently the points of intersection of the conics ω'^2_1 and ω''^2_1.

§ 4. As l cuts each of the four hyperboloids twice, the *image curve* has *double points* in S_k. As λ^6 is a rational curve it has 6 more double points; they are due to curves k^3 that cut l twice. Hence a line chosen at random is a *bisecant* of *six* curves of the congruence.

Two curves λ^6 have 20 points F in common; hence on two lines there rest 20 k^3 and the curves resting on l form a surface Λ^{20}. On this the lines b_k are *sextuple* and the ten transversals t'_k, t''_k are *quadruple* (l has four points in common with O'_4).

Two surfaces Λ have the lines b_k, t'_k, t''_k and 20 curves k^3 in common.

§ 5. The system Σ'_2 of the k^3 consisting of a k^2 and the transversal t'_2 of b_1, b_3, b_4, b_5, is represented on the points of the tangent c'_2 that corresponds to the point of intersection of b_1 and t'_2.

The systems Σ'_2 and Σ_3 have the degenerate k^3 in common consisting of t'_2, t'_3 and the line of intersection of the planes $b_2\, t'_3$ and $b_3\, t'_2$. This k_3 has the point $c'_2\, c'_3$ as image.

Each point $c'_k\, c'_l$, $c'_k\, c''_l$, $c''_k\, c''_l$ $(k, l = 2, 3, 4, 5)$ is the image of a k^3 consisting of three parts. Also the non singular points of intersection of c'_k (c''_k) with ω'^2_1 and ω''^2_1 are images of such composite k^3.

Hence Γ contains *fourty* k^3 consisting of *three straight lines*.

§ 6. Besides the hyperboloids considered in § 3 there are six more hyperboloids each of which is defined by three lines b, that contain ∞^1 k^3 of Γ; they are not represented in singular points.

Two hyperboloids have a k^3 in common only when they have a common directrix b. For H_{123} and H_{345} cut each other along b_3 and a k^3 that has b_3, hence all b_k, as bisecants.

As H_{345} is represented in S_2, H_{245} in S_3 and H_{123} has a k^3 in common with each of these two hyperboloids, the image of H_{123} passes through S_2 and S_3. Analogously the image of H_{145} contains the points S_4 and S_5.

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[1]) If the points of H_{234} are projected out of a point of this surface on a plane, the images of the k^3 form a pencil of rational curves c^3 that have their double point in one of the cardinal points of the representation. To them belong the images of four composite k^3.

As H_{123} has only one k^3 in common with H_{145}, their *images* are *straight lines*, viz. the lines $S_2 S_3$ and $S_4 S_5$; we shall indicate them by h_{123} and h_{145}.

Hence the *ten hyperboloids* are represented as carriers of curves of Γ in the points S_k and the sides of the *complete quadrilateral* defined by them.

§ 7. On the conic Ψ^2 which H_{123} has in common with a plane Ψ, the k^3 of H_{123} define an involution I^3. Hence H_{123} contains *four* k^3 that touch Ψ. Consequently the image curve of the system Ψ of the k^3 touching Ψ has *quadruple points* in S_k and cuts each line h in four more points; it is, therefore, a Ψ^{12} (S^4).

With $\omega'_1{}^2(S)$ it has eight points F in common that are images of curves k^3 on $O'_1{}^4$; in six of these composite k^3 the k^2 touches Ψ and the point of intersection of Ψ and t'_1 defines a figure which must be counted twice.

According to $\Psi^{12}(S^4)$ and $\lambda^6(S^2)$ Ψ is a surface of the degree 40. As a k^3 of Γ chosen at random has only points of b_k in common with Ψ, the lines b are *twelvefold* lines; the transversals t', t'' are evidently *eightfold* on Ψ.

§ 8. The *surface* $(C)^6$ of the k^3 that cut b_1 in a point C, is *represented* on the points of the line c (§ 1) corresponding to C. A line k^3 that does not lie on $(C)^6$ cuts b_1 in two points; the other 16 points of intersection lie on the other four b; these are, therefore, *double lines* of $(C)^6$.

The plane Cb_2 contains two k^2 that form two k^3 on $(C)^6$ together with t'_2 and t''_2; the tangents at C to these k^2 are not coplanar with b_1, hence C is a *double point* on $(C)^6$; the transversal through C of b_2 and b_3 cuts $(C)^6$ twice in C and twice on b_2 and on b_3.

As the image line c has two points in common with each of the conics $\omega'_1{}^2$ and $\omega''_1{}^2$, t'_1 and t''_1 are *double lines*; the other eight lines t', t'' are *single* on $(C)^6$.

§ 9. The surface $(B_2)^6$ of the k^3 that cut b_2 in the point B, has in common with $(C)^6$ the lines $b_1, b_2, t'_1, t''_1, t'_2, t''_2$, which must be counted double, the lines b_3, b_4, b_5, which must be counted four times, and the six lines t'_k, t''_k $(k=3, 4, 5)$; besides these lines they have two curves k^3 in common. Hence through *two points* chosen at random on b_k and b_l, there pass *two* curves of Γ.

Consequently the image curve of the system on $(B_k)^6$ is a *conic* $\beta_k{}^2$. The conics $\beta_k{}^2$ form a system with index 2 for the k^3 represented in a point F belongs to 2 points B_k.

As $(B_k)^6$ contains one k^3 of each of the systems Σ'_l, Σ''_l $(l \neq k, 1)$ (§ 5), $\beta_k{}^2$ passes through the three singular points S_l. The image curves $\beta_2^2(S_3 S_4 S_5)$ and $\beta_3^2(S_2 S_4 S_5)$ have two points F in common: this shows again that two surfaces $(B_k)^6$ and $(B_l)^6$ have two k^3 in common.

§ 10. A line d of the image plane is the image of a surface $\triangle^6$ that passes through b_1 and has the lines b_2, b_3, b_4, b_5 as *double lines*. For d cuts each line c in one point and each β_k^2 in two points.

Also t'_1, t''_1 are *double lines*, because each of the conics ω'^2 has two points in common with d.

The *field of rays* $[d]$ is, therefore, the image of a *net* of *surfaces* $\triangle^6$ with a basis consisting of *six* double lines and *nine* single lines (b_1 and eight lines t', t''). This figure together with each k^3 of Γ forms the basis of a pencil belonging to this net.

There are evidently *four* more similar nets.

§ 11. As a surface Λ^{20} has a sextuple line in b_1 (§ 4), the k^3 that cut a line m resting on b_1 in C outside C, form a surface M''; on this t'_1, t''^4 are *double lines*, the other 8 lines t are *triple lines*. (To the systems Σ there correspond surfaces O^4).

Any hyperboloid H that does not contain b_1, has two points in common with m; accordingly the image curve μ of M^{14} has double points in S_k, μ has three points F and the double point S_k in common with the image line c'_k or c''_k of a system Σ_k; it is, therefore, a $\mu^5\,(S^2)$.

It has 14 points F in common with $\lambda^6\,(S^2)$; this shows again that the degree of the surface M is 14.

A line c contains 5 points F of μ^5; hence b_1 is a *quintuple* line of M^{14}. The other lines b_k are *quadruple*; this appears when we consider the intersection of this surface and an arbitrary k^3. Combination of $\mu^5\,(S^2)$ with $\omega^2\,(S)$ and with c'_k (c''_k) gives as result that M^{14} has t'_1 and t''_1 as *double lines* and that the other 8 transversals are *triple* lines. When the line l rests on b_1, $\Lambda^{20}\,(b_k{}^6, t_k{}^4)$ decomposes into $(C)^6\,(b_1, b_k{}^2)$ and $M^{14}\,(b^1{}_5, b_k{}^4)$.

If l cuts two lines b, Λ^{20} is replaced by two surfaces $(B)^6$ and a surface $M^8\,(b_k{}^3\, b_l{}^3\, b_p{}^2\, b_q{}^2\, b_r{}^2)$.

§ 12. A ray n of the *plane pencil* (N, ν) is a *bisecant* of *six* ϱ^3 (§ 4). On the intersection of ν and a hyperboloid H the k^3 define an I^3; two of the lines containing the pairs of this I^3 pass through N. Hence H contains two k^3 of the system of the k^3 that cut a ray n twice.

The line h_{123} contains accordingly two points F of the image curve and the double points S_2, S_3; it is, therefore, a $\nu^6\,(S^2)$.

As $\nu^6\,(S^2)$ and $\lambda^6\,(S^2)$ have 20 points F in common, the k^3 of the system form a *surface* N^{20} with *sextuple* b_k and *quadruple* t_k; for ν^6 cuts a c 6 times, a c'_k 4 times.

§ 13. Let E be the system of the k^3 of which a *tangent* passes through the point N.

The tangents of the k^3 on H form a congruence [3, 4]. For through a point P of H there passes a line that is a bisecant of $\infty^1\, k^3$, hence a

tangent of two k^3, and the line that touches the k^3 through P at P. And as the aforesaid I^3 has four double points, a plane contains four tangents.

Consequently the image curve of E has *triple* points in S_k and as h_{123} also contains three points F this curve is an $\varepsilon^9 (S^3)$.

It has 30 points F in common with $\lambda^6 (S^2)$; accordingly the k^3 of the system E form a *surface* E^{30}. On this surface the lines b are *ninefold*, the lines t *sixfold*. As a check the fact may serve that E^{30} has 30 curves k^3 in common with Λ^{20} besides the multiple lines b and t.

§ 14. A conic ω^2 through the four points S is the image of a surface O^4, for $\omega^2 (S)$ has four points F in common with $\lambda^6 (S^2)$. As ω^2 has two points in common with any line c and one point F with any β^2, b_1 is double line and O^4 contains the four lines b_k $(k \neq 1)$. Also the eight lines t_k lie on O^4.

The *pencil* (ω^2) is, therefore, the image of a *pencil* (O^4) of which the basis consists of the double line b_1 and the 12 lines b_k and t_k.

Chemistry. — *Osmosis of ternary liquids. General considerations* V.
By F. A. H. SCHREINEMAKERS.

(Communicated at the meeting of April 28, 1928).

The diffusing mixture and the real membrane.

In the preceding communications (Gen. III and IV) we have seen in what way the composition of the diffused mixture can be found and how from this the directions in which the different substances pass through the membrane may be deduced, the composition of the diffused liquid, etc. We have seen that the composition of the diffused mixture, which we shall call L_0, is represented by the point of intersection S_0 of two conjugated chords. This is only the case, however, when the membrane itself does not contain the diffusing substances; as, however, the membrane does contain these substances, we shall call the first a "theoretical" and the second a "real" membrane.

In fig. 1 two points of the one branch of an osmosis-path are represented by 1 and 2 and two points of the other branch by 1′ and 2′. In point 1 (and 1′) of the path we imagine an osmotic system with a real membrane; we represent this by:

$$l_1 \times L_1 \mid m_1 \times M_1 \mid r_1 \times L'_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

On the left side of the membrane there are l_1 quantities of a liquid L_1 and on the right side r_1 quantities of a liquid L'_1. We represent the composition of L_1 by:

$$x_1 \text{ quant. of } X + y_1 \text{ quant. of } Y + (1 - x_1 - y_1) \text{ quant. of } W \quad . \quad (2)$$

and that of L'_1 by:

$$x'_1 \text{ quant. of } X + y'_1 \text{ quant. of } Y + (1 - x'_1 - y'_1) \text{ quant. of } W \quad . \quad (3)$$

Although the substances will not be equally spread in the membrane, yet we may say that it contains a definite quantity of liquid of a definite composition. We shall say that the membrane contains m_1 quantities of a liquid M_1, the composition of which we shall represent by:

$$\alpha_1 \text{ quant. of } X + \beta_1 \text{ quant. of } Y + (1 - \alpha_1 - \beta_1) \text{ quant. of } W \quad . \quad (4)$$

We imagine this liquid M_1 to be represented in fig. 1 by the point 1″. In the point 2 (and 2′) of the path we then have an osmotic system:

$$l_2 \times L_2 \mid m_2 \times M_2 \mid r_2 \times L'_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

In order to represent the compositions of these liquids we imagine in (2), (3) and (4) the index 1 to be substituted by 2. The liquid M_2 of the membrane has been represented in fig. 1 by point 2″.

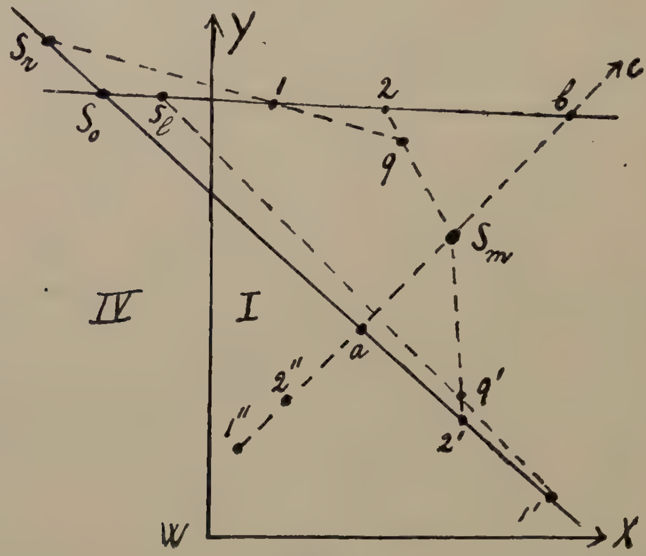

Fig. 1.

Consequently the osmosis-path of the system does not consist of 2, but of three branches; if namely the left side liquid travels along the part 1 . 2, and, therefore, the right side liquid along the part 1′ . 2′, then the membrane travels along the part 1″ . 2″ of the path.

If the left side liquid gives off a certain quantity u of a substance, then the right side liquid will take in a quantity, smaller or larger than u, according as the membrane will take in or give off a little of this substance during that time; it may even be supposed that as well the left side liquid as the right side liquid give e.g. the substance X to the membrane or take it from the membrane. For this reason we cannot speak any more of a single diffused mixture, but we shall distinguish three mixtures; we call them L_l, L_m and L_r and represent them in fig. 1 by the points s_l, s_m and s_r.

L_l is the mixture which is taken in or given off by the left side liquid. As L_1 has passed into L_2 by taking in or giving off a certain quantity of L_l, s_l, therefore, must be situated somewhere on the line 1 . 2. If s_l is situated in the way drawn in fig. 1, then this mixture has been given off by the left side liquid. We represent the composition of this mixture by:

$$x \text{ quant. of } X + y \text{ quant. of } Y + (1 - x - y) \text{ quant. of } W. \quad (6)$$

L_m is the mixture, which is taken in or given off by the membrane; consequently the point s_m is situated somewhere on the line 1″ . 2″. In

fig. 1 it has been assumed that this mixture has been taken in by the membrane. We represent its composition by:

α quant of $X + \beta$ quant. of $Y + (1 - \alpha - \beta)$ quant. of W . . (7)

L_r is the mixture which is taken in or given off by the right side liquid; consequently the point s_r is situated somewhere on the line $1'.2'$. In fig. 1 we have assumed that this mixture has been taken in by the right side liquid. We represent its composition by:

x' quant. of $X + y'$ quant. of $Y + (1 - x' - y')$ quant. of W . (8)

Now we shall assume that the left side liquid has given off l quantities of L_l and the membrane has taken in m quantities of L_m and the right side liquid r quantities of L_r.

It appears from the liquids L_1 and L_2 on the left side of the membrane that during the osmosis $l_1 - l_2$ quantities have disappeared here; they contain $l_1 x_1 - l_2 x_2$ quantities of X and $l_1 y_1 - l_2 y_2$ quantities of Y; of course the remainder is the substance W. Consequently we find:

$$l = l_1 - l_2 \quad x = \frac{l_1 x_1 - l_2 x_2}{l} \quad y = \frac{l_1 y_1 - l_2 y_2}{l} \quad . \; . \; . \quad (9)$$

From the liquids in the membrane and on the right side of it, follows:

$$m = m_2 - m_1 \quad \alpha = \frac{m_2 \alpha_2 - m_1 \alpha_1}{m} \quad \beta = \frac{m_2 \beta_2 - m_1 \beta_1}{m} \quad . \; . \quad (10)$$

$$r = r_2 - r_1 \quad x' = \frac{r_2 x'_2 - r_1 x'_1}{r} \quad y' = \frac{r_2 y'_2 - r_1 y'_1}{r} \quad . \; . \quad (11)$$

As the quantities of the substances do not change during the osmosis, we have:

$$\left.\begin{array}{l} l_1 + m_1 + r_1 = l_2 + m_2 + r_2 \\ l_1 x_1 + m_1 \alpha_1 + r_1 x'_1 = l_2 x_2 + m_2 \alpha_2 + r_2 x'_2 \\ l_1 y_1 + m_1 \beta_1 + r_1 y'_1 = l_2 y_2 + m_2 \beta_2 + r_2 y'_2 \end{array}\right\} \quad . \; . \; . \; . \quad (12)$$

With the aid of (9), (10) and (11) we find from this:

$$l = m + r \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (13)$$

$$l x = m \alpha + r x' \qquad l y = m \beta + r y'$$

If we consider the compositions of the mixtures L_l, L_m and L_r then it follows from (13):

$$l \times L_l = m \times L_m + r \times L_r \quad . \; . \; . \; . \; . \; . \; . \quad (14)$$

This expresses that the entire mixture, which has disappeared from the left side liquid, has been taken in by the membrane and the right side liquid. As this speaks for itself, we are able to write down (14) without any further deduction. It now follows from (14):

the point s_l is situated between s_m and s_r and divides the line into two parts, which are determined by:

$$s_l s_r : s_l s_m = m : r \quad . \; . \; . \; . \; . \; . \; . \quad (15)$$

We shall now assume that, besides the compositions of the left side and right side liquids, their quantities l_1, l_2, r_1 and r_2 are known also. Then we are able to determine l, x and y from (9) and r, x' and y' from (11). Next we find m, α and β from (13). Consequently we know the quantities and the compositions of the three diffused mixtures.

Matters are otherwise, however, if we only know the compositions of the left side and right side liquids. Then the point S_0 is indeed known but not the points S_l and S_r, which are essential in order to determine the directions in which the substances now pass through the membrane. Yet in many cases it is possible to find these directions with the aid of point S_0 only, as will be shown later on.

When system (1) has passed into (5), then the membrane has taken in m quantities of L_m. We now put these m quantities in the left side liquid L_2; this now changes its composition and passes again into a liquid L_q which has been represented in fig. 1 by point q. This liquid has been formed from l_2 quantities of L_2 and m quantities of L_m; consequently point q is situated on the line $2 . S_m$ and it divides this line into two parts, which are determined by:

$$q . 2 : q . s_m = m : l_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

So, instead of system (5) we now get the system:

$$(l_2 + m) \times L_q \mid m_1 \times M_1 \mid r_2 \times L'_2 \quad . \quad . \quad . \quad . \quad . \quad (17)$$

so that the membrane has not changed its composition in passing from (1) to (17); consequently the diffused mixture is represented by the point of intersection of the lines $1 . q$ and $1' . 2'$. As the state of things on the right side of the membrane in (17) is now the same as in (5), the mixture L_r must have diffused; consequently point s_r is the point of intersection of the lines $1 . q$ and $1' . 2'$.

If we put the m quantities of L_m in the right side liquid, then we get the system:

$$l_2 \times L_2 \mid m_1 \times M_1 \mid (r_2 + m) \times L'_q \quad . \quad . \quad . \quad . \quad . \quad (18)$$

The liquid L'_2 of (5) has now been replaced by a liquid L'_q which is situated in fig. 1 on the line $2' . s_m$; for this obtains:

$$q' . 2' : q' s_m = m : r_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

As the state of things on the left side of the membrane now is the same in (18) as in (5), the point of intersection of the lines $1 . 2$ and $1' . q'$ falls in the point s_l. Consequently we find:

s_l is the point of intersection of the chord $1 . 2$ with the line $1' . q'$
s_r is the point of intersection of the line $1 . q$ with the chord $1' . 2'$.

In fig. 1 s_r is situated on the left side and s_l on the right side of s_0; this is the case when s_m is situated between a and b; if, however, s_m

is situated between $2''$ and a, so that point q' comes on the other side of the chord $1' \cdot 2'$, then we see that both points come on the left side of s_0. If, however, s_m is situated between b and c (we imagine c at infinite distance) then both points fall on the right side of s_0.

In the transition-cases, viz. when s_m is situated in a or in b, then s_l or s_r coincide with s_0.

We are able to express the length of the lines $s_0 s_l$ and $s_0 s_r$ in different ways. A simple way of expressing it is among others:

$$s_0 s_l = \frac{m}{l} \cdot \beta \qquad s_0 s_r = -\frac{m}{r} \cdot \alpha \quad . \quad . \quad . \quad . \quad . \quad (20)$$

in which, however, α and β have another meaning than above, though they are connected with them. We are able to deduce these equations in different ways, one of which we shall briefly indicate.

It is namely possible to prove that the equations (13) obtain for each system of coordinates; we now choose the point s_0 as origin, the line $s_0 1'$ as X-axis and the line $s_0 1$ as Y-axis. Then α is the distance of s_m to the line $s_0 1$, measured along a line parallel to the line $s_0 1'$; β is the distance of s_m to the line $s_0 1'$, measured along a line parallel to $s_0 1$.

For the point s_l now obtains $x=0$ and $y=s_0 s_l$ and for point s_r we find $x'=s_0 s_r$ and $y'=0$. Substituting these values in (13) we find (20).

In fig. 1 α and β are both positive; now (20) gives a positive value for $s_0 s_l$ and a negative value for $s_0 s_r$; we see that this is in accordance with fig. 1.

From what precedes it appears that the points s_l and s_r can be situated in different ways with respect to s_0; it follows from (20) that the smaller m is with respect to l and r, the nearer they will be to s_0. For the present we shall leave it so; to the other case we are going to refer later on.

Above we have noticed already that we have to know the points s_l and s_r in order to determine the composition of the diffusing mixtures and the directions in which they and the substances pass through the membrane. In order to consider what mistakes may arise, if we use the point s_0 instead of these points, we shall discuss a few cases.

First we shall suppose the point s_0 to be in field IV not too close to one of the sides (or their prolongations) of the triangle, so that the points s_l and s_r will be situated in this field as well.

If we were to use the point s_0 now, then we should make a mistake in the determination of the mixtures L_l and L_r which really have diffused and of course this will always be the case, when the points do not coincide.

The directions, in which the three substances and their mixtures pass through the membrane, are, however, also indicated by the point s_0.

The same obtains also for other fields, except, as we shall see later on, towards the end of the osmosis.

Now we assume that during the osmosis point s_0 passes from field IV to field I; as in this case no divergences of the substances Y and W occur, we need only consider the substance X.

As long as s_0 is situated in field IV, the mixture contains a negative quantity of X; as liquid 1 gives off this mixture, a negative quantity of X will go towards the right; so the substance X will really go towards the left.

If s_0 comes on the side WY, so that the mixture does not contain X, then no X will consequently pass through the membrane.

If s_0 comes in field I then the mixture contains a positive quantity of X; now the substance X passes through the membrane towards the right. Point s_0 now yields the simple scheme:

$$\underset{\longleftarrow}{X} \qquad \underset{\text{———}}{X} \qquad \underset{\longrightarrow}{X} \quad \ldots\ldots\ldots \quad (21)$$

The first symbol obtains when s_0 is situated in field IV, the second when s_0 is situated in field I; the dash indicates that at that moment no X passes through the membrane.

Each of the points s_l and s_r indicates a similar scheme as (21); yet there is a difference viz. the transition-symbol occurs in these three schemes at different moments of the osmosis.

We now shall assume that the points are situated with respect to one another as shown in fig. 1; then point s_l will be the first to come on the side WY, next point s_0 and at last s_r. If the three systems are combined to form one, we get scheme (22).

$$\left.\begin{array}{l|lllll|l}
 & \;\;1 & \;\;2 & \;\;3 & \;\;4 & \;\;5 \quad\;\; 6 & \;\;7 \\
S_l \longleftarrow & \text{——} & \longrightarrow & \longrightarrow & \longrightarrow & \longrightarrow & \longrightarrow S_l \\
S_r \longleftarrow & \longleftarrow & \longleftarrow & \longleftarrow & \longleftarrow & \text{——} & \longrightarrow S_r \\
S_0 \longleftarrow & \longleftarrow & \longleftarrow & \text{——} & \longrightarrow & \longrightarrow & \longrightarrow S_0
\end{array}\right\} \quad \ldots \quad (22)$$

The top row (at the beginning and the end of which s_l has been placed) represents the s_l-scheme; we call these arrows and directions the s_l-arrows and s_l-directions. Consequently these arrows indicate what has really been happening to the left side liquid. Therefore, an arrow pointing to the left indicates that the left side liquid takes in X, an arrow pointing to the right that this left side liquid gives off X.

The middle row is the s_r-scheme. The s_r-arrows indicate, therefore, what really happens to the right side liquid. Consequently an arrow pointing to the left indicates that the right side liquid gives off X, an arrow pointing to the right that this right side liquid takes in X.

The lowest row is the s_0-scheme; consequently the s_0-arrows only indicate what can be deduced from point s_0; now we must consider in how far they are in accordance with reality.

For this purpose we divide (18) into seven groups.

Group 1 obtains as long as the three points are still situated in field IV.

Group 2 obtains when s_l comes on the side WY.

Group 3 obtains when s_l is in field I but the other points still in IV.

Group 4 obtains when s_0 comes on the side WY.

Group 5 obtains when s_r is still in IV, but the other points already in I.

Group 6 obtains when s_r comes on the side WY.

Group 7 obtains when the three points are situated in field I.

From this now appears:

in group 1 and 7, so when the three points are situated at the same time either in field IV or in field I, the s_0-arrow indicates the real direction, in which the substance X passes through the membrane; in the transition-groups 2—6 this is no more the case.

Let us e.g. take group 2. It appears from the s_l-dash that the left side liquid does not absorb or give off X at that moment; it follows from the s_r-arrow that the right side liquid gives off X; consequently this has been absorbed by the membrane.

The s_0-arrow, therefore, does not completely concur with reality; for it indicates that X from the right side liquid has gone towards the left side liquid; in reality, however, X has gone from the right side liquid towards the membrane.

It is also clear that the s_0-arrows cannot fully represent the state of things in the other transition-groups either; in each of these groups the s_l- and s_r-arrows namely have contrary direction; consequently they cannot be replaced by a single arrow.

Further consideration of what is really happening in the groups 3—6 is left to the reader.

So we find the following:

the s_0-scheme corresponds with reality as long as point s_0 does not approach one of the sides of the triangle too much; in the vicinity of side WY, however, it obtains no more for the substance X and we have to replace it by the s_l- and s_r-schemes. In general the quantities of X duffusing during these transitions, are small.

Of course the same things obtain for the substances Y or W when point s_0 arrives in the vicinity of the sides WX or XY.

Previously (Gen. III fig. 3) we have already seen that point s_0 can also pass from field IV towards VII during the osmosis; this is the case when the chords $1 . 2$ and $1' . 2'$ (fig. 1) become parallel.

As long as s_0 is situated in field IV we have the scheme:

$$\underset{\longleftarrow}{X} \quad \underset{\longrightarrow}{Y} \quad \underset{\longrightarrow}{W} \quad \ldots\ldots\ldots \quad (23)$$

Then namely the left side liquid gives off a mixture s_0, which contains a negative quantity of X but a positive quantity of Y and W.

When s_0 comes in field VII, then the left side liquid absorbs this mixture; this, however, now contains a positive quantity of X and a negative quantity of Y and W; consequently we again get scheme (23).

At the moment that s_0 passes from field IV towards VII, scheme (23) obtains as well as we have previously (Gen. III) seen. Remarkable is only that at this moment as much of X diffuses towards the left as Y and W towards the right.

Consequently we get the scheme (23) no matter whether s_0 is situated in field IV or VII; as this also obtains for the points s_r and s_l we find, therefore:

when point s_0 passes from field IV towards VII; then the s_0-scheme indicates the directions in which the substances really pass through the membrane.

It is easy to see, however, that this is no more the case for the directions, in which the mixtures L_l and L_r pass through the membrane. For this another scheme with five transition-groups may be deduced; in these cases, however, the quantities of the diffusing mixtures are generally small. The deduction of these schemes I leave to the reader.

In a system with a theoretical membrane the composition of the mixture which really passes through the membrane, is represented by s_0; this point, therefore, can never be situated as shown in figs. 2 or 3.

In fig. 2 namely s_0 is situated between the points 1 and 2 and it is clear that L_1 can never pass into L_2 by absorbing or giving off this mixture s_0. Consequently the point s_0 cannot be situated between $1'$ and $2'$ either. Of course s_0 cannot be situated as drawn in fig. 3, for then L_1 as well as L_2 should have to absorb this mixture.

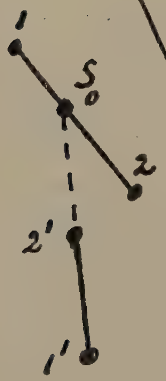

Fig. 2.

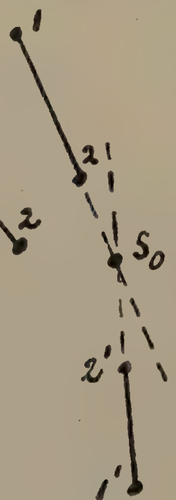

Fig. 3.

In a system with a real membrane the mixtures, really absorbed or given off on the left and on the right side, are represented by s_l and s_r. It is clear now that s_l, which must be situated on the chord $1\,.\,2$, can never be situated between the points 1 and 2; the point s_r which must be situated on the chord $1'\,.\,2'$ can never be situated between the points $1'$ and $2'$. Although in a system with a theoretical membrane the point s_0 cannot be situated as drawn in figs. 2 and 3, this is actually the case in a system with a real membrane, especially when the points 2 and $2'$ draw nearer to each other.

We imagine that system (1) approaches its final condition without

changing the composition of its membrane; then the liquids become equal on both sides of the membrane. We represent this system by:

$$l_e \times L_e \,\vdots\, m_1 \times M_1 \,\vdots\, r_e \times L_e \quad \ldots\ldots \quad (24)$$

The final liquid l is now represented in fig. 4 by a point e on the

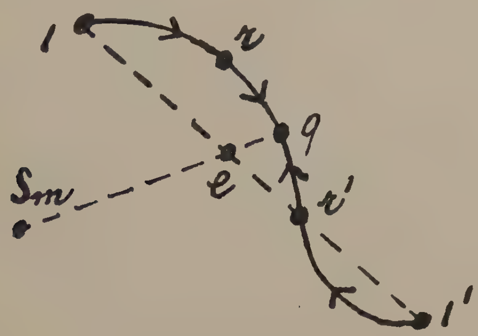

Fig. 4.

line $1 \, . \, 1'$. As, however, the membrane has absorbed m quantities of a mixture L_m, the system (24) will not come into existence in reality, but a system:

$$l_q \times L_q \,\vdots\, m_q \times M_q \,\vdots\, r_q \times L_q \quad \ldots\ldots \quad (25)$$

This liquid L_q has another composition than L_e and has been represented in fig. 4 by point q. If we represent the composition of L_m by the point s_m, then q must be situated on the prolongation of the line $s_m e$. We now find:

$$eq = \frac{m}{r_1 + l_1 - m} \times e S_m \quad \ldots\ldots \quad (26)$$

Consequently the position of the point q depends on the quantity of m and the composition of the mixture L_m which has been absorbed by the membrane. From this appears among other things that even the position of point q depends on the dimensions of the membrane.

Instead of a path with the final point e on the line $1 \, . \, 1'$ we consequently get a path, which has been shifted a little and has its point in q.

Branch $1' . q$ of this path now intersects the line $1 \, . \, 1'$ in a point r'; consequently a point r conjugated with r' must be situated on branch $1 \, . \, q$.

If the liquid 2 is situated between r and q consequently $2'$ between r' and q then we see that the point of intersection s_0 of the chords $1 \, . \, 2$ and $1' . \, 2'$ is situated as drawn in fig. 2.

If branch $1' . q$ did not intersect the line $1 \, . \, 1'$ then s_0 would be situated as drawn in fig. 3.

Consequently we see that towards the end of the osmosis point s_0

can be situated as drawn in the figs. 2 and 3; it is clear that in this case no s_0-scheme can be formed any more.

So the following things appear among others from our considerations.

If we only know the compositions of the liquids on the left and the right side of the membrane, then we can only find the point s_0; this gives only approximated values for s_l en s_r.

In general the s_0-scheme indicates the exact directions, in which the different substances and the mixtures L_l and L_r go through the membrane; it is, however, no longer absolutely valid for any of the substances, when the point s_0 comes in the vicinity of one of the sides of the triangle (e.g. for the substance X in the vicinity of the side WY).

neither does it obtain absolutely any more for the direction of the diffusing mixture when s_0 is situated at infinite distance (e.g. with the transition from field IV towards field VII).

It is of no use towards the end of the osmosis.

If, however, not only the compositions, but also the quantities of the liquids on the left and the right side of the membrane are known, then we can find L_l and L_r and they enable us to determine accurately what has been happening during the osmosis.

(To be continued.)

Leiden, Lab. of Inorganic. Chemistry.

Mathematics. — *Invarianten der Integranden vielfacher Integrale in der Variationsrechnung.* II.*) By Prof. L. KOSCHMIEDER. (Communicated by Prof. R. WEITZENBÖCK.)

(Communicated at the meeting of November 26, 1927).

II. *Invarianten der Grundfunktion F.*

Alles folgende bezieht sich auf F (14) als Grundfunktion; in § 3 wurden deren Besonderheiten gegenüber (3) erörtert und dem weiteren dienliche Bezeichnungen eingeführt.

1. *Die erste Variation.*

§ 6. *Die einfachsten Invarianten.*

Wir beginnen mit der sogleich zu benutzenden Bemerkung, dass die Determinante

$$\xi = \begin{vmatrix} \xi_1 & \xi_2 & \dots & \xi_{n+1} \\ x_{1,1} & x_{2,1} & \dots & x_{n+1,1} \\ \dots & \dots & \dots & \dots \\ x_{1,n} & x_{2,n} & \dots & x_{n+1,n} \end{vmatrix} = \theta_i \xi_i, \quad \dots \dots \quad (40)$$

in der die ξ_i den in § 3, (18) erklärten Sinn haben, nach (18), (20) eine Punktinvariante vom Gewichte -1 ist,

$$\xi' = D^{-1}\,\xi \quad \dots \dots \dots \quad (41)$$

Betrachten wir den Schnitt $x_i = \grave{x}_i$ der Überfläche $\mathfrak{f}$ (2) und einer andern $\grave{\mathfrak{f}}$ mit der Parameterdarstellung $\grave{x}_i = \grave{\varphi}_i(\grave{u}_\alpha)$! Wir bilden bei dieser die Ableitungen $\grave{x}_{i,\alpha} = \partial\grave{\varphi}_i/\partial\grave{u}_\alpha$ und bezeichnen die den θ_k entsprechenden Determinanten der Matrix $\|\grave{x}_{i,\alpha}\|$ mit $\grave{\theta}_k$; dann ist nach (41) die Grösse $\grave{\theta}_i\,\xi_i$ punktinvariant vom Gewichte -1. Schreiben wir gemäss (16) kurz $\Gamma(x_i, \theta_i) = F$, $\Gamma(\grave{x}_i, \grave{\theta}_i) = \grave{F}$, so finden wir, indem wir nach dem beim Beweise des Satzes 1 gesagten die ξ_i durch die $\partial F/\partial\theta_i$ (21) ersetzen, den Ausdruck

$$\mathfrak{T}_{\mathfrak{f},\grave{\mathfrak{f}}} = \frac{\partial F}{\partial \theta_i}\,\grave{\theta}_i$$

als eine absolute Punktinvariante [34]). Nun ist, wenn $\mathfrak{f}$ eine Extremale des

*) I. in diesen Proceedings, Vol. **31**, N⁰. 2, (1928), S. 140.

[34]) Man kann dies sofort aus (20), (21) entnehmen; uns lag an dem Zusammenhange mit (40).

Variationsproblems $\delta I = 0$ ist, $\mathfrak{T}_{\mathfrak{f},\grave{\mathfrak{f}}}$ die Grösse, deren Verschwinden die *Transversalität* [35]) von $\mathfrak{f}$ zu $\grave{\mathfrak{f}}$ aussagt; daher erweist sich diese Beziehung als invariant. Da sich ferner bei derselben Aufgabe die *E-Funktion* in der Form darstellen lässt [36])

$$E(x_i, \theta_i, \grave{\theta}_i) = \grave{F} - \mathfrak{T}_{\mathfrak{f},\grave{\mathfrak{f}}},$$

so ist wegen der Invarianz von F und $\mathfrak{T}$ auch E absolut invariant.

Mit Rücksicht darauf, dass die Variationen δx_i (25) Veränderliche der Art ξ_i (18) sind, entnimmt man aus (40), (41) weiter, dass die Grösse

$$v = \theta_i \, \delta x_i, \quad . \quad . \quad . \quad (42) \qquad\qquad v' = D^{-1} v \quad . \quad . \quad . \quad (43)$$

eine Punktinvariante vom Gewichte —1 ist.

Wir wenden uns jetzt zur *Variation der Grundfunktion*; es ist

$$\delta F = \frac{\partial F}{\partial x_i} \delta x_i + \frac{\partial F}{\partial x_{i,\alpha}} \delta x_{i,\alpha} = \frac{\partial}{\partial u_\alpha}\left(\frac{\partial F}{\partial x_{i,\alpha}} \delta x_i\right) + W\theta_i \, \delta x_i, \quad . \quad (44)$$

wo die Ausdrücke $W\theta_i = W_i$ die Variationsableitungen [1]), (7), (8) bedeuten. Nach (19) und Satz 3 ist δF absolut invariant,

$$\delta F' = \delta F \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (45)$$

Da dies ferner nach der in (9) enthaltenen Formel

$$\frac{\partial F}{\partial x_{i,\alpha}} = \frac{\partial F'}{\partial x'_{k,\alpha}} \frac{\partial x'_k}{\partial x_i} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (46)$$

und nach (25. 1) für die Klammer im ersten Gliede

$$t = \frac{\partial}{\partial u_\alpha}\left(\frac{\partial F}{\partial x_{i,\alpha}} \delta x_i\right) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (47)$$

auf der rechten Seite von (44) und daher nach Satz 2 für dieses selbst gilt, so ist dort auch das zweite Glied vW absolut invariant,

$$v' W' = vW \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (48)$$

Nach (43) ist mithin W eine Punktinvariante vom Gewichte 1,

$$W' = DW \; {}^{37)} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (49)$$

Auf demselben Wege [38]) kann man auch die Parameterinvarianz der Funktion W dartun. Zunächst ist nach (37) und Satz 4

$$\delta \bar{F} = \mathfrak{D} \, \delta F \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (50)$$

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[35]) Nach RADON [20]), S. 58.

[36]) Ebd. S. 60.

[37]) Dieser kurze Beweis ist dem von BOLZA [9]), S. 349 bei einfachen Integralen gegebenen nachgebildet.

[38]) In anderer Weise leitet RADON [20]), S. 57 die Invarianz (57) her. — A. a. O. [1]), S. 189 habe ich (49), (57) aus der ausdrücklichen Darstellung der W_i entwickelt.

In der Formel (44)

$$\delta F = t + vW \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

nimmt die Grösse t (47) bei dem Wechsel (5) gleichfalls den Faktor $\mathfrak{D}$ an. Denn es ist nach [1]), $\overline{(31)}$

$$\left.\begin{aligned}\frac{\partial \bar{F}}{\partial x_{i,\bar{\alpha}}} &= \frac{\partial F}{\partial x_{i,\lambda}}\,\mathfrak{D}\,\frac{\partial \bar{u}_\alpha}{\partial u_\lambda},\\ \bar{t} &= \mathfrak{D}\,\frac{\partial}{\partial u_\lambda}\left(\frac{\partial F}{\partial x_{i,\lambda}}\,\delta x_i\right) + \frac{\partial F}{\partial x_{i,\lambda}}\,\delta x_i\,\frac{\partial}{\partial \bar{u}_\alpha}\left(\mathfrak{D}\,\frac{\partial \bar{u}_\alpha}{\partial u_\lambda}\right)\end{aligned}\right\}; \quad . \quad . \quad (52)$$

infolge der bekannten Beziehung

$$\frac{\partial}{\partial \bar{u}_\alpha}\left(\mathfrak{D}\,\frac{\partial \bar{u}_\alpha}{\partial u_\lambda}\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53)$$

bleibt

$$\bar{t} = \mathfrak{D} t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54)$$

Aus (50), (54) schliesst man mit Hilfe von (51), dass auch

$$\bar{v}\,\overline{W} = \mathfrak{D} v W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55)$$

ist; da sich nach (42), (38), (36. 1)

$$\bar{v} = \mathfrak{D} v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

ergibt, erkennt man W als absolute Parameterinvariante,

$$\overline{W} = W \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (57)$$

§ 7. *Die Funktion F_1 von* De Donder.

Es sei auf $\mathfrak{f}$ (2)

$$\theta^2 \equiv \theta_i \theta_i > 0 \text{ [39])} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (58)$$

Wir setzen

$$\frac{1}{\theta^2}\,\frac{\partial^2 F}{\partial x_{i,\alpha}\,\partial x_{i,\beta}} = \Phi_{\alpha\beta} \text{ [40])}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

$$\text{det. } \Phi_{\alpha\beta} = \Phi \;, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

Zunächst bringen wir die Determinante Φ zu der Funktion F_1 von De Donder[5]) in Beziehung. Aus der Formel

$$\frac{\partial F}{\partial x_{i,\alpha}} = \frac{\partial F}{\partial \theta_l}\,\theta_{l,i\alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61)$$

---

[39]) Vgl. [1]), S. 181; $\theta > 0$.

[40]) $\Phi_{\alpha\beta}$ ist in [1]), (5) mit $F_{\alpha\beta}/\theta^2$, in [6]), (17) mit $F^*_{\alpha\beta}$ bezeichnet. Φ (60) hat in der Schreibweise [6]), S. 141 den Wert $\Phi\theta^{-2n}$.

in der $\theta_{l,i\alpha} = -\theta_{i,l\alpha}$ die algebraische Ergänzung des Elementes $x_{i,\alpha}$ bezüglich der Determinante θ_l bedeutet $(\theta_{i,i\alpha}$[41]$) = 0)$, erhalten wir

$$\frac{\partial^2 F}{\partial x_{i,\alpha}\, \partial x_{p,\beta}} = \frac{\partial^2 F}{\partial \theta_l\, \partial \theta_m}\, \theta_{l,i\alpha}\, \theta_{m,p\beta} - \frac{\partial F}{\partial \theta_l}\, \frac{\partial \theta_{i,l\alpha}}{\partial x_{p,\beta}}.$$

Ist $p = i$, so fällt der Subtrahend rechts fort, weil $x_{i,\alpha}$ in θ_i nicht auftritt (§ 3); multipliziert man diese einfachere Gleichung mit $x_{h,\alpha}\, x_{k,\beta}$, wendet die elementaren Determinantenformeln

$$\theta_{p,q\gamma}\, x_{r,\gamma} = j_{qr}\, \theta_p - j_{rp}\, \theta_q, \quad j_{lm} = \begin{cases} 0 \text{ für } l \neq m, \\ 1 \text{ für } l = m \end{cases}$$

an und beachtet die aus (17) durch Differentiation folgende Beziehung

$$\theta_l\, \frac{\partial^2 F}{\partial \theta_l\, \partial \theta_m} = 0, \text{ [42]}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (62)$$

so findet man nach Summation über i

$$\Phi_{\alpha\beta}\, x_{h,\alpha}\, x_{k,\beta} = \frac{\partial^2 F}{\partial \theta_h\, \partial \theta_k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (63)$$

In der $(n+1)$-reihigen Determinante der Grössen rechts hat die algebraische Ergänzung f_{hk} des Elementes $\partial^2 F / \partial \theta_h\, \partial \theta_k$ nach (63) den Wert

$$f_{hk} = (-1)^{h+k} \operatorname{det.} (\Phi_{\alpha\beta}\, x_{H,\alpha}\, x_{K,\beta}),$$

wo H und K die Zahlen von 1 bis $n+1$ mit Ausnahme von h und k vorstellen; da hier die rechte Seite sich in das Produkt $\Phi\, \theta_h\, \theta_k$ umformen lässt, stimmt die Determinante Φ in der Tat mit der von De Donder durch die Formel

$$f_{hk} \equiv \operatorname{adj.} \frac{\partial^2 F}{\partial \theta_h\, \partial \theta_k} = F_1\, \theta_h\, \theta_k \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

erklärten Funktion F_1 überein, *es ist*

$$\Phi = F_1 \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

Was ferner die Transformation der Ausdrücke $\Phi_{\alpha\beta}$, Φ betrifft, so ergibt sich aus [1]), (20), (23) und der nach u. (38), (58) geltenden Gleichung

$$\bar{\theta} = \mathfrak{D}\theta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

dass

$$\Phi'_{\alpha\beta} = D^2\, \Phi_{\alpha\beta}, \quad . \quad . \quad . \quad (67) \qquad \bar{\Phi}_{\alpha\beta} = \mathfrak{D}^{-1}\, \Phi_{\lambda\mu}\, \frac{\partial \bar{u}_\alpha}{\partial u_\lambda}\, \frac{\partial \bar{u}_\beta}{\partial u_\mu}, \quad . \quad . \quad . \quad . \quad (68)$$

$$\Phi' = D^{2n}\, \Phi \text{ [43]}), \quad . \quad . \quad . \quad (69) \qquad \bar{\Phi} = \mathfrak{D}^{-n-2}\, \Phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (70)$$

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[41]) Hier wird über i nicht summiert.

[42]) Vgl. Radon [20]), S. 55. Die dortigen Grössen $\Phi_{\alpha\beta}$ und unsere $\Phi_{\alpha\beta}$ (59) erweisen sich auf Grund von (63) und [20]), (7) als gleich.

[43]) Nach (65) ist also $F_1' = D^{2n} F_1$; nicht $F_1' = D^2 F_1$, wie a. a. O. [5]) irrtümlich angegeben.

ist. Wir setzen Φ wie auch F in dem betrachteten Gebiete [44]) etwa als positiv voraus; dieses Vorzeichen bleibt bei den Verwandlungen (69), (70) und (19), (37) erhalten.

Mit Hilfe der vorstehenden Formeln gewinnt man die Transformation mehrerer später wichtiger Grössen. Nach (67), (69) ist

$$\Phi'_{\alpha 3}\, \Phi'^{-\frac{1}{n}} = \Phi_{\alpha 3}\, \Phi^{-\frac{1}{n}}\,; \quad \ldots \ldots \quad (71)$$

hier wie im folgenden verstehen wir unter der mehrdeutigen Potenz deren reellen positiven Wert. Setzt man weiter

$$F^{n+2}\, \Phi = f, \quad \ldots \ldots \quad (72)$$

so findet man aus (19), (37), (69), (70)

$$f' = D^{2n} f, \quad \ldots \quad (73) \qquad \bar{f} = f \quad \ldots \quad (74)$$

Die Ausdrücke

$$\vartheta_i = f^{\frac{1}{2n}}\, \theta_i \quad \ldots \ldots \quad (75)$$

verwandeln sich auf Grund von (20), (38), (73), (74) wie folgt:

$$\vartheta'_k = \frac{\partial x_i}{\partial x'_k}\, \vartheta_i, \quad \ldots \quad (76) \qquad \bar{\vartheta}_i = \mathfrak{D}\, \vartheta_i \quad \ldots \quad (77)$$

Vermöge (69) leitet man aus W (49) die absolute Punktinvariante

$$M = W \Phi^{-\frac{1}{2n}}, \quad \ldots \quad (78) \qquad M' = M \text{ [45])} \quad \ldots \quad (79)$$

her; darüber hinaus zeigen (73), (74), dass die Grösse

$$S = W f^{-\frac{1}{2n}} = W\, (F^{n+2}\, \Phi)^{-\frac{1}{2n}} \quad \ldots \ldots \quad (80)$$

eine absolute Punkt- und Parameterinvariante ist [46]),

$$S' = S, \quad \ldots \quad (81) \qquad \bar{S} = S \quad \ldots \quad (82)$$

§ 8. *Das Grundintegral als Anlass zu einer Massbestimmung.*

P. FINSLER [47]) und L. BERWALD [48]) haben die Differentialgeometrie eines N-stufigen Raumes aufgebaut, in der auf jeder Kurve die Bogenlänge

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[44]) Dieses ist hinsichtlich der x_i der Bereich X_{n+1} [s. (1)]; die $x_{i,\varkappa}$ haben beliebige endliche Werte von der Art, dass (58) gilt.

[45]) Diese Punktinvariante hat DE DONDER a. a. O. [5]) gefunden; die dort mitgeteilte Gestalt $WF_1^{-\frac{1}{2}}$ ist nach [43]) durch $WF_1^{-\frac{1}{2n}}$ zu ersetzen.

[46]) Wir nannten $-S$ a. a. O. [1]), (14) die "mittlere extremale (Ueberflächen-) Krümmung".

[47]) Ueber Kurven und Flächen in allgemeinen Räumen; Dissertation, Göttingen 1918.

[48]) *a)* Jahresber. d. Deutsch. Math.-Ver. **34** (1925), S. 213—220; *b)* Math. Zeitschr. **25** (1926), S. 40—73; *c)* Lotos (Prag) **74** (1926), S. 43—51; *d)* Journ. f. d. reine u. angew. Math. 156 (1927), S. 191—222.

durch das einfache Grundintegral eines Variationsproblems erster Ordnung [49]) erklärt ist. Sieht man in ähnlicher Weise bei jeder im $n+1$-stufigen Raume X_{n+1} gelegenen Überfläche X_n (2) das Integral (15) als Mass ihres Inhaltes an, so ermöglicht dieses n-fache Grundintegral, wie ich im folgenden andeuten will, eine *Massbestimmung in* X_{n+1}. BERWALD bildet seine Entwickelungen in der Form dem Sonderfalle nach, in dem das Integral die Bogenlänge in einem RIEMANNschen Raume $\mathfrak{X}_N$ angibt [50]); entsprechend vergleichen wir hier das allgemeine Integral (15) mit demjenigen besonderen, welches den Inhalt $\mathfrak{J}$ eines Teiles der in einen RIEMANNschen Raum $\mathfrak{X}_{n+1}$ gebetteten Überfläche $\mathfrak{X}_n$ (2) darstellt. Das Linienelement in $\mathfrak{X}_{n+1}$ sei $d\mathfrak{s}^2 = \mathfrak{a}_{ik}\, dx_i\, dx_k$; die $\mathfrak{a}_{ik}$ sind Funktionen der x_l allein. Mit Hilfe ihrer algebraischen Ergänzungen $\mathfrak{A}_{ik}$ bezüglich der Determinante $\mathfrak{a}$ der $\mathfrak{a}_{ik}$ drückt $\mathfrak{J}$ sich in der Gestalt aus

$$\mathfrak{J} = \int^{(n)} \mathfrak{F}\, du, \quad . \quad . \quad (83) \qquad\qquad \mathfrak{F}^2 = \mathfrak{A}_{ik}\, \theta_i \theta_k \; {}^{51)}; \quad . \quad . \quad (84)$$

ersichtlich besteht die Beziehung

$$\mathfrak{A}_{ik} = \tfrac{1}{2} \frac{\partial^2 \mathfrak{F}^2}{\partial\theta_i\, \partial\theta_k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85)$$

Demgemäss führt man bei dem Integrale (15) zur Erklärung eines Grundtensors in dem *allgemeinen Raume* X_{n+1} zunächst die Grössen

$$A_{ik} = \tfrac{1}{2} \frac{\partial^2 F^2}{\partial\theta_i\, \partial\theta_k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86)$$

ein; entsprechend (84) gilt dann die Formel

$$F^2 = A_{ik}\, \theta_i\, \theta_k\,, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

da F (17) in den θ_i homogen von erster Stufe ist. Mit Ausnahme des Falles (84), in dem die $\mathfrak{A}_{ik}$ reine Ortsfunktionen sind, hängen die A_{ik} ausser von den x_l auch von den θ_l ab.

Die Determinante

$$A = \text{det.}\, A_{ik} = \text{det.} \left(\frac{\partial F}{\partial\theta_i} \frac{\partial F}{\partial\theta_k} + F \frac{\partial^2 F}{\partial\theta_i\, \partial\theta_k} \right)$$

lässt sich leicht berechnen. Schreibt man sie nämlich als Summe von 2^{n+1} Determinanten, deren Elemente teils von der Gestalt $\sigma_{ik} = \frac{\partial F}{\partial\theta_i} \frac{\partial F}{\partial\theta_k}$ sind, teils die Form $\tau_{ik} = F \frac{\partial^2 F}{\partial\theta_i\, \partial\theta_k}$ besitzen, so verschwinden von diesen Determinanten alle, bei denen mindestens zwei Spalten Elemente σ_{ik} enthalten; ferner diejenige, welche aus lauter Elementen τ_{ik} besteht, weil dies nach (64) von ihrer Reziproken $F^{n(n+1)}$ det. f_{ik}

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[49]) $n = 1$, $q = 1$ in der Bezeichnung (3).

[50]) Vgl. [47]) a), S. 216.

[51]) Vgl. z. B. [2]), S. 253.

gilt. Es bleiben die Determinanten zu summieren, bei denen die σ_{ik} genau in einer Spalte auftreten; da dann die algebraischen Ergänzungen der σ_{ik} die Werte $F^n f_{ik}$ haben, ergibt sich nach (64), (65), (17), (72)

$$A = \frac{\partial F}{\partial \theta_i} \frac{\partial F}{\partial \theta_k} F^n \Phi \theta_i \theta_k = F^{n+2} \Phi = f \quad . \quad . \quad . \quad . \quad (88)$$

Demnach ist $A \neq 0$. Unter Heranziehung der Grössen (75) kann man also die Invariante F^2 (87) in der Form $A^{-\frac{1}{n}} A_{ik} \vartheta_i \vartheta_k$ darstellen; dabei bilden nach (76) die ϑ_i einen bei dem Wechsel (1) kovarianten Vektor. Folglich sind die Ausdrücke $a^{ik} = A^{-\frac{1}{n}} A_{ik}$ die kontravarianten Komponenten eines Tensors zweiter Stufe; die diesen Sachverhalt ausdrückenden Formeln

$$a'^{lm} = a^{ik} \frac{\partial x'_l}{\partial x_i} \frac{\partial x'_m}{\partial x_k}$$

sind an Hand von (86), (88), (20), (73) leicht auch unmittelbar zu bestätigen. Wir sehen den — übrigens parameterinvarianten — Tensor der a^{ik} als den massbestimmenden *Grundtensor* in X_{n+1} an; seine kovarianten Komponenten sind die mit $A^{\frac{1}{n}-1}$ multiplizierten algebraischen Ergänzungen der Elemente A_{ik} in Bezug auf A. Die a_{ik} sind Funktionen der x_l und — abgesehen von dem Sonderfalle (84) — der θ_l; es ist det. $a_{ik} = A^{\frac{1}{n}}$. Wir erklären jetzt das *Linienelement* in X_{n+1} durch die Formel

$$ds^2 = a_{ik}\, dx_i\, dx_k.$$

Weitere differentialgeometrische Ausführungen in der durch das Vorstehende bezeichneten Richtung werde ich an anderer Stelle folgen lassen.

II. *Die zweite Variation.*

§ 9. *Die Verallgemeinerung der Transformation von* UNDERHILL.

Die zweite Variation der Grundfunktion F ist nach (51)

$$\delta^2 F = \delta t + W \delta v + v\, \delta W, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (89)$$

Da aus (42) durch Variation und Multiplikation mit W die Formel

$$W \delta v = W \delta \theta_i\, \delta x_i + W \theta_i\, \delta^2 x_i \quad . \quad . \quad . \quad . \quad . \quad . \quad (90)$$

hervorgeht und ferner gemäss [1]), (8)

$$\delta W_i\, \delta x_i = v\, \delta W + W \delta \theta_i\, \delta x_i$$

ist, ergibt sich

$$W \delta v + v \delta W = \delta W_i\, \delta x_i + W_i\, \delta^2 x_i, \quad . \quad . \quad . \quad . \quad . \quad (91)$$

$$\delta^2 F = \delta t + \delta W_i\, \delta x_i + W_i\, \delta^2 x_i. \quad . \quad . \quad . \quad . \quad . \quad (92)$$

Der zweite Summand rechts hat, wie aus der a.a.O. [6]), § 1 durchgeführten Rechnung [vgl. dort die Formeln (12), . . . , (15)] ohne Aenderung übernommen werden kann, den Wert

$$\delta W_i\ \delta x_i = \frac{\partial W_i}{\partial x_k}\ \delta x_i\ \delta x_k - F_{ik,\alpha\beta}\ \delta x_i\ \delta x_{k,\alpha\beta}$$
$$+\left(\frac{\partial^2 F}{\partial x_i\ \partial x_{k,\alpha}} - \frac{\partial^2 F}{\partial x_{i,\alpha}\partial x_k} - \frac{\partial}{\partial u_\beta}\ \frac{\partial^2 F}{\partial x_{i,\beta}\partial x_{k,\alpha}}\right)\delta x_i\ \delta x_{k,\alpha}\,;$$

dem Zeichen $F_{ik,\alpha\beta}$ kommt dabei die a.a.O. [6]), S. 133 bzw. [1]), S. 188 angegebene Bedeutung zu. Schreibt man

$$-F_{ik,\alpha\beta}\ \delta x_i\ \delta x_{k,\alpha\beta} = \frac{\partial F_{ik,\alpha\beta}}{\partial u_\beta}\ \delta x_i\ \delta x_{k,\alpha} - \delta x_i\ \frac{\partial}{\partial u_\beta}\left(F_{ik,\alpha\beta}\ \delta x_{k,\alpha}\right)$$

und zieht den Ausdruck [6]), (16) bzw. [1]), (30) der Vierzeigergrössen heran [40]), so wird weiter

$$\left.\begin{aligned} \delta W_i\ \delta x_i &= \frac{\partial W_i}{\partial x_k}\ \delta x_i\ \delta x_k - \delta x_i\ \frac{\partial}{\partial u_\beta}\left(\Phi_{\alpha\beta}\ \theta_i\ \theta_k\ \delta x_{k,\alpha}\right)\\ &+\left[\frac{\partial^2 F}{\partial x_i\ \partial x_{k,\alpha}} - \frac{\partial^2 F}{\partial x_{i,\alpha}\ \partial x_k} + \frac{1}{2}\ \frac{\partial}{\partial u_\beta}\left(\frac{\partial^2 F}{\partial x_{i,\alpha}\partial x_{k,\beta}} - \frac{\partial^2 F}{\partial x_{i,\beta}\ \partial x_{k,\alpha}}\right)\right]\delta x_i\ \delta x_{k,\alpha}. \end{aligned}\right\}\quad(93)$$

Hier greift nun die *Hilfsformel*

$$\left.\begin{aligned} &\frac{\partial^2 F}{\partial x_i\ \partial x_{k,\alpha}} - \frac{\partial^2 F}{\partial x_{i,\alpha}\ \partial x_k} + \frac{1}{2}\ \frac{\partial}{\partial u_\beta}\left(\frac{\partial^2 F}{\partial x_{i,\alpha}\partial x_{k,\beta}} - \frac{\partial^2 F}{\partial x_{i,\beta}\ \partial x_{k,\alpha}}\right)\\ &= \theta_{i,k\alpha}\ W + \Phi_{\alpha\beta}\left(\frac{\partial\theta_i}{\partial u_\beta}\ \theta_k - \frac{\partial\theta_k}{\partial u_\beta}\ \theta_i\right) \end{aligned}\right\}\quad.\quad(94)$$

ein. Sie ist a.a.O. [6]), § 2, (28) im Falle einer Extremale hergeleitet, d.h. unter der Annahme $W = 0$, von der wir uns hier freimachen. Die Beziehungen [6]), (20), . . . , (26) sind von letzterer unabhängig. Auf der rechten Seite von [6]), (26) ersetzt man das erste Glied nach [1]), (7) durch den Ausdruck $\theta_{k,i\alpha}\left(\frac{\partial F}{\partial x_k} - W_k\right)$ [52]) ; die weitere Rechnung unterscheidet sich also von der a.a.O. [6]) angestellten lediglich dadurch, dass man dort rechts das Glied $\theta_{i,k\alpha}\ W_k$ [52]) hinzufügt. Statt [6]), (28) erhält man mithin (94).

Trägt man (94) in (93) ein, beachtet, dass dann rechts $\theta_{i,k\alpha}\ \delta x_{k,\alpha} = \delta\theta_i$ ist, und addiert beiderseits $W\theta_i\ \delta^2 x_i$, so findet man laut (91), (90) mit Rücksicht auf [1]), (8), auf (42) und die Symmetrie der Grössen (59)

$$v\,\delta W + W\delta v = \frac{\partial W_i}{\partial x_k}\ \delta x_i\ \delta x_k - v\ \frac{\partial}{\partial u_\beta}\left(\Phi_{\alpha\beta}\ \theta_k\ \delta x_{k,\alpha}\right)$$
$$- v\Phi_{\alpha\beta}\ \frac{\partial\theta_k}{\partial u_\beta}\ \delta x_{k,\alpha} + W\delta v,$$
$$v\,\delta W = \frac{\partial W_i}{\partial x_k}\ \delta x_i\ \delta x_k - v\ \frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial v}{\partial u_\beta}\right) + v\ \delta x_k\ \frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\ \frac{\partial\theta_k}{\partial u_\beta}\right).$$

[52]) Hier ist nach k nicht zu summieren.

Wenn man daher

$$\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial \theta_i}{\partial u_3}\right)=P_i \text{ [53]}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

$$\tfrac{1}{2}\left(\frac{\partial W_i}{\partial x_k}+\frac{\partial W_k}{\partial x_i}+P_k\,\theta_i+P_i\,\theta_k\right)=L_{ik}=L_{ki} \text{ [54]}) \quad . \quad . \quad (96)$$

setzt, so wird

$$v\,\delta W=L_{ik}\,\delta x_i\,\delta x_k-v\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial v}{\partial u_3}\right) \quad . \quad . \quad . \quad . \quad (97)$$

Da bei Einführung der Bezeichnung

$$\frac{\partial W}{\partial x_i}+P_i=Q_i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

nach [1]), (8) die Formeln gelten

$$L_{ik}=\tfrac{1}{2}(Q_i\,\theta_k+Q_k\,\theta_i),\quad L_{ik}\,\delta x_i\,\delta x_k=vQ_i\,\delta x_i, \quad . \quad . \quad . \quad (99)$$

entnimmt man aus (97) als Wert der Variation von W

$$\delta W=Q_i\,\delta x_i-\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial v}{\partial u_3}\right) \quad . \quad . \quad . \quad . \quad . \quad (100)$$

Durch Einsetzung der Werte (97) bzw. (100) in (89) gewinnt man diejenigen beiden *Darstellungen der Grösse $\delta^2 F$, welche* UNDERHILLS auf den Fall $n=1$ bezügliche *Transformation der zweiten Variation* [8]) *auf unser Grundintegral* (15) *verallgemeinern.*

Wir zerlegen in § 10 den als Bestandteil von $\delta^2 F$ auftretenden Ausdruck

$$\delta(vW)=W\delta v-v\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial v}{\partial u_\beta}\right)+vQ_i\,\delta x_i, \quad . \quad . \quad . \quad (101)$$

der nach (48) und Satz 3 eine absolute Punktinvariante ist, in weitere invariante Summanden.

§ 10. *Die Punktinvariante Ψ.*

Es werde in (101) rechterhand statt v (42), (43) die laut (43), (69) absolut punktinvariante Grösse

$$\Omega=\Phi^{\frac{1}{2n}}v, \quad . \quad . \quad (102) \qquad\qquad \Omega'=\Omega \quad . \quad . \quad . \quad (103)$$

eingeführt. Aus $v=\Phi^{-\frac{1}{2n}}\Omega$ folgt

$$\delta v=\Phi^{-\frac{1}{2n}}\delta\Omega-\frac{1}{2n}\,\Omega\Phi^{-\frac{1}{2n}}\frac{\delta\Phi}{\Phi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (104)$$

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[53]) P_i sind die a. a. O. [6]), (33) mit A_i bezeichneten Ausdrücke.

[54]) Ist $n=1$, so stimmen die Grössen L_{11}, $L_{12}=L_{21}$, L_{22} mit den von UNDERHILL [3]), S. 327 eingeführten L_1, M_1, N_1 überein.

und es ist

$$\frac{\delta\Phi}{\Phi}=\frac{1}{\Phi}\frac{\partial\Phi}{\partial x_i}\delta x_i+\frac{\partial}{\partial u_\alpha}\left(\frac{1}{\Phi}\frac{\partial\Phi}{\partial x_{i,\alpha}}\delta x_i\right)-\delta x_i\frac{\partial}{\partial u_\alpha}\left(\frac{1}{\Phi}\frac{\partial\Phi}{\partial x_{i,\alpha}}\right) \quad . \quad (105)$$

Dabei kann man die Ausdrücke $\partial\Phi/\partial x_{i,\alpha}$ auch nach Art der Grössen (61) im Sinne von (65) geschrieben denken. Das zweite Glied auf der rechten Seite von (101) bringt man leicht auf die Form

$$v\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial v}{\partial u_\beta}\right)=\Omega\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\,\Phi^{-\frac{1}{n}}\frac{\partial\Omega}{\partial u_\beta}\right)+\Omega^2\,\Phi^{-\frac{1}{2n}}\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial\Phi^{-\frac{1}{2n}}}{\partial u_\beta}\right);(106)$$

man erhält daher statt (101)

$$\left.\begin{aligned}\delta(vW)&=W\Phi^{-\frac{1}{2n}}\delta\Omega-\frac{1}{2n}\Omega W\Phi^{-\frac{1}{2n}}\frac{\partial}{\partial u_\alpha}\left(\frac{1}{\Phi}\frac{\partial\Phi}{\partial x_{i,\alpha}}\delta x_i\right)\\&-\Omega\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\Phi^{-\frac{1}{n}}\frac{\partial\Omega}{\partial u_\beta}\right)-\frac{1}{2n}\Omega W\Phi^{-\frac{1}{2n}}\left[\frac{1}{\Phi}\frac{\partial\Phi}{\partial x_i}-\frac{\partial}{\partial u_\alpha}\left(\frac{1}{\Phi}\frac{\partial\Phi}{\partial x_{i,\alpha}}\right)\right]\delta x_i\\&+\Omega\Phi^{-\frac{1}{2n}}Q_i\,\delta x_i-\Omega^2\Phi^{-\frac{1}{2n}}\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial\Phi^{-\frac{1}{2n}}}{\partial u_\beta}\right).\end{aligned}\right\}(107)$$

Wir werden jetzt zeigen, dass in dieser, wie erwähnt absolut invarianten Verbindung jedes der drei ersten Glieder $\mathfrak{Z}_1, \mathfrak{Z}_2, \mathfrak{Z}_3$ rechts die Invarianzeigenschaft besitzt; ist dies nachgewiesen, so ist eine neue Invariante ermittelt, nämlich die Summe $\mathfrak{Z}$ der drei letzten Glieder auf der rechten Seite von (107).

Die Invarianz $\mathfrak{Z}_1'=\mathfrak{Z}_1$ folgt aus (79), (103) und Satz 3. Was $\mathfrak{Z}_2$ betrifft, so ist nach (69)

$$\frac{\partial\Phi'}{\partial x'_{i,\alpha}}=D^{2n}\frac{\partial\Phi}{\partial x_{k,\alpha}}\frac{\partial x_k}{\partial x'_i},$$

sodass durch Division mit (69) und Zusammensetzung mit den $\delta x'_i=\delta x_l\,.\,\partial x'_i/\partial x_l$ sich ergibt

$$\frac{1}{\Phi'}\frac{\partial\Phi'}{\partial x'_{i,\alpha}}\delta x'_i=\frac{1}{\Phi}\frac{\partial\Phi}{\partial x_{i,\alpha}}\delta x_i; \quad . \; . \; . \; . \; . \; . \quad (108)$$

mit Hilfe des Satzes 2 und kraft (79), (103) bestätigt man, dass in der Tat $\mathfrak{Z}_2'=\mathfrak{Z}_2$ ist. In $\mathfrak{Z}_3$ ist nach (71) und Satz 2 zunächst der Inhalt der Klammer invariant, daher auch $\mathfrak{Z}_3=\mathfrak{Z}_3'$.

Nach der an (107) angeschlossenen Bemerkung ist jetzt die Invarianz $\mathfrak{Z}'=\mathfrak{Z}$ dargetan. Mit $\mathfrak{Z}$ ist auch die Grösse $\mathfrak{Z}/\Omega=\Psi$, also *der Ausdruck*

$$\left.\begin{aligned}\Psi(x_i, x_{i,\alpha}, x_{i,\alpha\beta}, x_{i,\alpha\beta\gamma}, \delta x_i)&=\frac{1}{2n}W\Phi^{-\frac{1}{2n}}\left[-\frac{1}{\Phi}\frac{\partial\Phi}{\partial x_i}+\frac{\partial}{\partial u_\alpha}\left(\frac{1}{\Phi}\frac{\partial\Phi}{\partial x_{i,\alpha}}\right)\right]\delta x_i\\&+\Phi^{-\frac{1}{2n}}Q_i\,\delta x_i-\Omega\,\Phi^{-\frac{1}{2n}}\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial\Phi^{-\frac{1}{2n}}}{\partial u_\beta}\right)\end{aligned}\right\}(109)$$

absolut punktinvariant.

§ 11. *Die Punktinvariante U.*

Da man die Variationen δx_i (25) als Veränderliche ξ_i (18) auffassen kann, treffen auf die Invariante (109) die Voraussetzungen des Satzes 1 zu. Ersetzt man daher in Ψ die δx_i durch die Werte $\partial F/\partial \theta_i$ und somit Ω (102) gemäss (42), (17) durch $\Phi^{\frac{1}{2n}} F$, so entsteht eine Invariante der Art g vom Gewichte 1, aus der nach Multiplikation mit $\Phi^{-\frac{1}{2n}}$ zufolge (69) eine absolute Invariante hervorgeht. Indem wir diese mit $-FU$ bezeichnen, *gelangen wir* bei irgendeiner Überfläche $\mathfrak{f}$ (2) *zu der absoluten Punktinvariante*

$$\left.\begin{aligned} U(x_i, x_{i.\alpha}, x_{i.\alpha\beta}, x_{i.\alpha\beta\gamma}) = &- F^{-1}\,\Phi^{-\frac{1}{n}} Q_i \frac{\partial F}{\partial \theta_i} + \Phi^{-\frac{1}{2n}} \frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial \Phi^{-\frac{1}{2n}}}{\partial u_\beta}\right) \\ &+ \frac{1}{2n} F^{-1}\, W \Phi^{-\frac{1}{n}} \frac{\partial F}{\partial \theta_i}\left[\frac{1}{\Phi}\frac{\partial \Phi}{\partial x_i} - \frac{\partial}{\partial u_\alpha}\left(\frac{1}{\Phi}\frac{\partial \Phi}{\partial x_{i.\alpha}}\right)\right], \end{aligned}\right\} \quad (110)$$

der Verallgemeinerung der Invariante K von UNDERHILL [55]), die sich aus U für $n=1$ ergibt. Das erste Glied $\mathfrak{Q}$ auf der rechten Seite von (110) kann man nach (17) und (99) auch schreiben

$$\mathfrak{Q} = - F^{-2}\Phi^{-\frac{1}{n}} L_{ik} \frac{\partial F}{\partial \theta_i}\frac{\partial F}{\partial \theta_k}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (111)$$

Ist $\mathfrak{f}$ im besonderen eine *Extremale*, also $W=0$, so lassen sich die Grössen L_{ik} durch eine einzige L darstellen in der Form [56])

$$L^*_{ik} = L\theta_i\theta_k \;{}^{57)}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (112)$$

daher vereinfacht sich die Invariante U zu

$$U^* = - F^{-1}\,\Phi^{-\frac{1}{n}} Q^*_i \frac{\partial F}{\partial \theta_i} + \Phi^{-\frac{1}{2n}} \frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial \Phi^{-\frac{1}{2n}}}{\partial u_\beta}\right) \quad . \quad . \quad (113)$$

bzw.
$$U^* = - \Phi^{-\frac{1}{n}} L + \Phi^{-\frac{1}{2n}} \frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial \Phi^{-\frac{1}{2n}}}{\partial u_\beta}\right) \quad . \quad . \quad . \quad (114)$$

Hiernach ist die Funktion L [6]), (41) der Gestalt fähig

$$L = F^{-1}\, Q^*_i \frac{\partial F}{\partial \theta_i}.$$

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55) A. a. O. [3]), S. 330.

56) Das Zeichen * deutet an, dass die mit ihm versehenen Grössen für eine Extremale zu berechnen sind.

57) A. a. O. [6]), S. 138. — Die Grösse L verallgemeinert den von K. WEIERSTRASS für $n=1$ mit F_2 [vgl. [9]), S. 226] bezeichneten Ausdruck.

§ 12. *Die Punkt- und Parameterinvariante* Ψ_0.

Um über die Punktinvariante Ψ (109) hinaus zu einem Ausdrucke fortzuschreiten, der ausserdem parameterinvariant ist, variieren wir die Grösse S (80). Diese ist absolut invariant in beiderlei Sinne (81), (82); dieselbe Eigenschaft hat nach den Sätzen 3 und 4 der Ausdruck

$$\delta S = S\left[\frac{\delta W}{W} - \frac{1}{2n}\frac{\delta \Phi}{\Phi} - \left(\frac{1}{2}+\frac{1}{n}\right)\frac{\delta F}{F}\right].$$

Da das laut (19), (45), (37), (50) auch von der Grösse $\delta F/F$ gilt, ist

$$\zeta = \frac{\delta W}{W} - \frac{1}{2n}\frac{\delta \Phi}{\Phi}$$

eine absolute Punkt- und Parameterinvariante, wie man mit Hilfe von (49), (57), (69), (70) auch unmittelbar feststellt.

Zu geeigneter Darstellung von ζ wendet man auf den Minuenden die Formel (100) an; man bedient sich an Stelle von v (42), (56) des Ausdrucks

$$V = F^{-\frac{1}{2}+\frac{1}{n}}\Phi^{\frac{1}{2n}} v = F^{-\frac{1}{2}+\frac{1}{n}}\Omega, \quad \ldots\ldots \quad (115)$$

der sich vermöge (103), (70) als absolut invariant in beiderlei Sinne erweist,

$$V' = V, \quad \ldots \quad (116) \qquad\qquad \overline{V} = V. \quad \ldots\ldots \quad (117)$$

Indem man bei der Umrechnung von (100) die den Faktor V enthaltenden Glieder zusammenfasst und den Subtrahenden in ζ ähnlich wie in (105) umformt, findet man

$$\left.\begin{aligned}\zeta = W^{-1} Q_i\,\delta x_i - F^{-\frac{1}{2}-\frac{1}{n}}\Phi^{-\frac{1}{2n}} S^{-1} V \frac{\partial}{\partial u_\alpha}\left[\Phi_{\alpha\beta}\frac{\partial}{\partial u_\beta}\left(F^{\frac{1}{2}-\frac{1}{n}}\Phi^{-\frac{1}{2n}}\right)\right]\\ -\frac{1}{2n}\frac{1}{\Phi}\frac{\partial \Phi}{\partial x_i}\delta x_i + \frac{1}{2n}\delta x_i \frac{1}{F}\frac{\partial}{\partial u_\alpha}\left(\frac{F}{\Phi}\frac{\partial \Phi}{\partial x_{i,\alpha}}\right)\\ -\frac{1}{FS}\frac{\partial}{\partial u_\alpha}\left(F^{1-\frac{2}{n}}\Phi^{-\frac{1}{n}}\Phi_{\alpha\beta}\frac{\partial V}{\partial u_\beta}\right) - \frac{1}{2n}\frac{1}{F}\frac{\partial}{\partial u_\alpha}\left(\frac{F}{\Phi}\frac{\partial \Phi}{\partial x_{i,\alpha}}\delta x_i\right).\end{aligned}\right\} \quad (118)$$

Nun ist hier rechts sowohl das vorletzte Glied $-\mathfrak{z}_4$ als auch das letzte $-\mathfrak{z}_5/2n$ invariant in beiderlei Sinne. Zunächst ist nämlich $\mathfrak{z}_4' = \mathfrak{z}_4$ infolge der Beziehungen (71), (116) und des Satzes 2. Ferner ergibt sich im Hinblick auf (37), (70), (68), (117), (82)

$$\bar{\mathfrak{z}}_4 = \mathfrak{D}^{-1} F^{-1} S^{-1} \frac{\partial}{\partial \bar{u}_\alpha}\left(\mathfrak{D} F^{1-\frac{2}{n}}\Phi^{-\frac{1}{n}}\Phi_{\lambda\mu}\frac{\partial \bar{u}_\alpha}{\partial u_\lambda}\frac{\partial \bar{u}_\beta}{\partial u_\mu}\frac{\partial V}{\partial u_\nu}\frac{\partial u_\nu}{\partial \bar{u}_\beta}\right);$$

benutzt man rechts die (in $u, \bar{u}$ statt in x, x' geschriebenen) Formeln [1]), (27) und differenziert die Klammer als ein Produkt, dessen einer Faktor

$\mathfrak{D} \cdot \partial \bar{u}_\alpha / \partial u_\lambda$ ist, so erhält man kraft (53) $\bar{\mathfrak{z}}_4 = \mathfrak{z}_4$. Aus (19), (108) und Satz 2 folgt $\mathfrak{z}_5' = \mathfrak{z}_5$; da nach (70)

$$\frac{1}{\bar{\Phi}} \frac{\partial \bar{\Phi}}{\partial x_{i,\bar{\alpha}}} = \frac{1}{\Phi} \frac{\partial \Phi}{\partial x_{i,\lambda}} \frac{\partial \bar{u}_\alpha}{\partial u_\lambda}$$

ist, stellt sich auf Grund von (36) und (53) auch

$$\bar{\mathfrak{z}}_5 = \frac{1}{\mathfrak{D} F} \frac{\partial}{\partial u_\alpha} \left(\mathfrak{D} \frac{\partial \bar{u}_\alpha}{\partial u_\lambda} \frac{F}{\Phi} \frac{\partial \Phi}{\partial x_{i,\lambda}} \delta x_i \right) = \mathfrak{z}_5$$

heraus. Die Invarianten $-\mathfrak{z}_4$ und $-\mathfrak{z}_5/2n$ trennt man von der Invariante ζ (118) ab; indem man deren verbleibenden Bestandteil mit S multipliziert und

$$\frac{S}{W} Q_i - \frac{1}{2n} \frac{S}{\Phi} \frac{\partial \Phi}{\partial x_i} + \frac{1}{2n} \frac{S}{F} \frac{\partial}{\partial u_\alpha} \left(\frac{F}{\Phi} \frac{\partial \Phi}{\partial x_{i,\alpha}} \right) = R_i$$

setzt, gewinnt man die gewünschte *absolute Invariante in beiderlei Sinne*

$$\Psi_0 (x_i, x_{i,\alpha}, x_{i,\alpha\beta}, x_{i,\alpha\beta\gamma}, \delta x_i) = R_i \, \delta x_i$$

$$- V F^{-\frac{2}{n}} \Phi^{-\frac{1}{2n}} \frac{\partial}{\partial u_\alpha} \left(\Phi_{\alpha\beta} \frac{\partial \Phi^{\frac{1}{2n}}}{\partial u_\beta} \right) - V F^{-\frac{1}{2}-\frac{1}{n}} \frac{\partial}{\partial u_\alpha} \left(\Phi^{-\frac{1}{n}} \Phi_{\alpha\beta} \frac{\partial F^{\frac{1}{2}-\frac{1}{n}}}{\partial u_\beta} \right).$$

Durch die vorgenommene Auflösung des zweiten Gliedes auf der rechten Seite von (118) in die beiden mit dem Faktor V behafteten Glieder von Ψ_0 wird die Beziehung dieses Ausdrucks zu Ψ (109) ersichtlich:

$$\left.\begin{aligned} \Psi_0 = F^{-\frac{1}{2}-\frac{1}{n}} \Psi - V F^{-\frac{1}{2}-\frac{1}{n}} \frac{\partial}{\partial u_\alpha} \left(\Phi^{-\frac{1}{n}} \Phi_{\alpha\beta} \frac{\partial F^{\frac{1}{2}-\frac{1}{n}}}{\partial u_\beta} \right) \\ + \frac{1}{2n} W F^{-\frac{3}{2}-\frac{1}{n}} \Phi^{-1-\frac{1}{2n}} \frac{\partial F}{\partial u_\alpha} \frac{\partial \Phi}{\partial x_{i,\alpha}} \delta x_i \,. \end{aligned}\right\} \quad (119)$$

§ 13. *Die Punkt- und Parameterinvariante U_0.*

Auf die Invariante Ψ_0, in der von den Veränderlichen der zweiten Reihe nur die absolut parameterinvarianten $\delta x_i = \xi_i$ auftreten, und zwar linear homogen, lassen sich die Sätze 1,6 anwenden. Wenn man demgemäss in Ψ_0 die δx_i durch die Grössen $\partial F / \partial \theta_i$ und daher V (115) durch $f^{\frac{1}{2n}}$ (72) ersetzt, erhält man aus Ψ_0 einen Ausdruck T, der punktinvariant vom Gewichte 1 und absolut parameterinvariant ist. Infolge von (73), (74) wird der Quotient $-T/f^{\frac{1}{2n}}$, also *die Grösse*

$$\left.\begin{aligned} U_0 (x_i, x_{i,\alpha}, x_{i,\alpha\beta}, x_{i,\alpha\beta\gamma}) = U_0 = F^{-\frac{2}{n}} U \\ + F^{-\frac{1}{2}-\frac{1}{n}} \frac{\partial}{\partial u_\alpha} \left(\Phi^{-\frac{1}{n}} \Phi_{\alpha\beta} \frac{\partial F^{\frac{1}{2}-\frac{1}{n}}}{\partial u_\beta} \right) - \frac{1}{2n} W F^{-2-\frac{2}{n}} \Phi^{-1-\frac{1}{n}} \frac{\partial F}{\partial u_\alpha} \frac{\partial F}{\partial \theta_i} \frac{\partial \Phi}{\partial x_{i,\alpha}} \end{aligned}\right\} (120)$$

eine absolute Punkt- und Parameterinvariante. Sie verallgemeinert die von UNDERHILL *bei einfachen Integralen angegebene Invariante* K_0 [58]), die aus U_0 für $n=1$ hervorgeht.

Für eine *Extremale* hat U_0 den Wert

$$U_0^* = F^{-\frac{2}{n}} U^* + F^{-\frac{1}{2}-\frac{1}{n}} \frac{\partial}{\partial u_\alpha}\left(\Phi^{-\frac{1}{n}} \Phi_{\alpha\beta} \frac{\partial F^{\frac{1}{2}-\frac{1}{n}}}{\partial u_\beta}\right) \quad . \quad . \quad (121)$$

oder, wie sich auf Grund von (114) nach kurzer Umrechnung ergibt,

$$U_0^* = -F^{-\frac{2}{n}} \Phi^{-\frac{1}{n}} L + F^{-\frac{1}{2}-\frac{1}{n}} \Phi^{-\frac{1}{2n}} \frac{\partial}{\partial u_\alpha}\left[\Phi_{\alpha\beta} \frac{\partial}{\partial u_\beta}\left(F^{\frac{1}{2}-\frac{1}{n}} \Phi^{-\frac{1}{2n}}\right)\right]. (122)$$

Auf die geometrische Bedeutung von U_0 [59]) bei RIEMANNschen $\mathfrak{X}_{n+1}$ und allgemeinen Räumen X_{n+1} (§ 8) werde ich, wie gesagt, bei anderer Gelegenheit eingehen. Hier sei noch die *übersichtliche Form* vermerkt, die man *der Grösse* $\delta^2 I$ bei einer festberandeten *Extremale* $\mathfrak{f}^*$ mit Hilfe von U^*, U_0^* geben kann, indem man gewissen von UNDERHILL für $n=1$ aufgestellten Formeln [60]) entsprechende beim Integrale (15) nachbildet.

Man bringt unter den genannten Annahmen die zweite Variation

$$\delta^2 I^* = \int^{(n)} \delta^2 F\, du = \int^{(n)} v\, \delta W\, du \quad ^{61)}$$

an Hand von (107), (99), (112) leicht auf die Gestalt

$$\delta^2 I^* = -\int^{(n)} \Omega \left[\frac{\partial}{\partial u_\alpha}\left(\Phi^{-\frac{1}{n}} \Phi_{\alpha\beta} \frac{\partial \Omega}{\partial u_\beta}\right) + U^* \Omega\right] du.$$

Die Einführung des Ausdrucks V (115) an Stelle von Ω liefert in Verbindung mit (121)

$$\delta^2 I^* = -\int^{(n)} V \left[\frac{\partial}{\partial u_\alpha}\left(F^{1-\frac{2}{n}} \Phi^{-\frac{1}{n}} \Phi_{\alpha\beta} \frac{\partial V}{\partial u_\beta}\right) + U_0^*\ V\ F\right] du;$$

durch teilweise Integration des ersten Gliedes rechts erhält man, indem man das Verschwinden von V längs des Randes berücksichtigt,

$$\delta^2 I^* = \int^{(n)} \left(F^{-\frac{2}{n}} \Phi^{-\frac{1}{n}} \Phi_{\alpha\beta} \frac{\partial V}{\partial u_\alpha} \frac{\partial V}{\partial u_\beta} - U_0^*\ V^2\right) F du.$$

Diese Formel gibt Anlass zu einer Bemerkung [62]) über das Vorzeichen

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58) A. o. O. 3), S. 334.

59) Diejenige von K_0 ist von UNDERHILL 3), S. 338 angegeben. — Die Invarianten K 55), K_0 von UNDERHILL treten in den neuesten Arbeiten über die Differentialgeometrie des Variationsproblems der Art 3) auf: BERWALD 48*b*), S. 61; *c*), S. 46, 52; *d*), S. 192.

60) A. a. O. 3), S. 335 f.

61) Vgl. u. (89).

62) Sie rührt für $n=1$ von UNDERHILL her; 3), S. 336.

von $\delta^2 I^*$: Ist die quadratische Form $\mathfrak{q} = \Phi_{\alpha\beta}\, w_\alpha\, w_\beta$ etwa definit positiv [63]), und ist $U_0^* < 0$ auf $\mathfrak{f}^*$, so gilt dort $\delta^2 I^* > 0$.

III. *Zerlegung der zweiten Variation in parameterinvariante Summanden.*

Bei der Darstellung (89) der zweiten Variation $\delta^2 F$, die nach (37) und Satz 4 eine Parameterinvariante [64]) vom Gewichte 1 ist, haben gemäss (54), (56), (57) die drei Glieder rechts die gleiche Eigenschaft,

$$\overline{\delta t} = \mathfrak{D}\delta t, \quad \overline{W\delta v} = \mathfrak{D} W\delta v, \quad \overline{v\delta W} = \mathfrak{D} v\,\delta W \quad . \quad . \quad . \quad (123)$$

Wir werden jetzt das dritte Glied $v\,\delta W$ weiter in fünf invariante Summanden zerspalten. Dabei nehmen wir in der a.a.O. [6]), S. 141 geschilderten Weise auf die invariante quadratische Differentialform $q_{\alpha\beta}\, du_\alpha\, du_\beta$ [65]) Bezug, deren Grundtensor die kontravarianten Komponenten

$$q^{\alpha\beta} = \theta\,\Phi_{\alpha\beta}$$

besitzt [40]). Wie dort bedienen wir uns der — bei invarianten χ, ψ gleichfalls invarianten — Differentiatoren

$$\nabla(\chi,\psi) = q^{\alpha\beta}\frac{\partial\chi}{\partial u_\alpha}\frac{\partial\psi}{\partial u_\beta}, \quad \Delta_1(\chi) = \nabla(\chi,\chi), \quad \Delta(\chi) = \frac{1}{\theta}\frac{\partial}{\partial u_\alpha}\left(\theta q^{\alpha\beta}\frac{\partial\chi}{\partial u_\beta}\right).$$

Mit diesen Zeichen kann man unter Verwendung der nach (56), (66) absolut invarianten Grösse

$$\omega = \theta^{-1} v, \quad . \quad . \quad (124) \qquad\qquad \overline{\omega} = \omega \quad . \quad . \quad . \quad (125)$$

in dem Ausdrucke [s. (101)]

$$v\,\delta W = -\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\, v\frac{\partial v}{\partial u_\beta}\right) + \Phi_{\alpha\beta}\frac{\partial v}{\partial u_\alpha}\frac{\partial v}{\partial u_\beta} + vQ_i\,\delta x_i$$

das erste und zweite Glied rechts schreiben

$$-\frac{1}{2}\frac{\partial}{\partial u_\alpha}\left(\Phi_{\alpha\beta}\frac{\partial v^2}{\partial u_\beta}\right) = -\frac{1}{2}\theta\,\Delta(\omega^2) - 2\omega\,\nabla(\theta,\omega) + \omega^2\left[\theta^{-1}\Delta_1(\theta) - \Delta(\theta)\right],$$

$$\Phi_{\alpha\beta}\frac{\partial v}{\partial u_\alpha}\frac{\partial v}{\partial u_\beta} = \theta^{-1}\Delta_1(\theta\omega) = \theta\Delta_1(\omega) + 2\,\omega\,\nabla(\theta,\omega) + \omega^2\theta^{-1}\Delta_1(\theta) \quad \text{[66])}.$$

Setzt man für Q_i den Wert (98) ein, fügt den Summanden $\omega^2 P_i\theta_i$ hinzu und zieht ihn in der Gestalt $\omega\theta^{-1}\,\theta_k P_k\,\theta_i\,\delta x_i$ wieder ab, so erhält man

$$\left.\begin{aligned} v\,\delta W = &-\frac{1}{2}\theta\Delta(\omega^2) + \theta\Delta_1(\omega) + \omega^2\left[2\,\theta^{-1}\Delta_1(\theta) - \Delta(\theta) + P_i\theta_i\right] \\ &+ \theta\omega\frac{\partial W}{\partial x_i}\,\delta x_i + \omega\theta^{-1}\,\theta_k\,(P_i\,\theta_k - P_k\,\theta_i)\,\delta x_i. \end{aligned}\right\} \quad (126)$$

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63) Der Faktor von $\mathfrak{q}$ ist nach § 7 positiv.

64) Da in III ausschliesslich von Parameterinvarianten die Rede ist, nennen wir diese weiterhin kurz Invarianten.

65) Die a. a. O. mit $g_{\alpha\beta}$ bezeichneten Grössen nennen wir hier $q_{\alpha\beta}$.

66) Vgl. [6]), (51).

Von den Gliedern rechts ist nach (66) und (125) das erste und zweite, nach (57), Satz 5 und (36 . 1) das vierte invariant vom Gewichte 1. Das dritte Glied hat dieselbe Eigenschaft; da nämlich darin der Faktor Θ von ω^2 gemäss [6]), (68) die Umformung gestattet

$$\Theta = 2\theta^{-1} \triangle_1(\theta) - \triangle(\theta) + P_i \theta_i = \theta^{-1} [\triangle_1(\theta) - \sum_i \triangle_1(\theta_i)], \quad (127)$$

so lehrt [6]), (58), dass $\overline{\Theta} = \mathfrak{D}\Theta$. Jetzt schliesst man aus (126) und (123.3), dass auch das fünfte Glied invariant vom Gewichte 1 ist; man überzeugt sich davon auch unmittelbar, indem man in ihm laut (95)

$$P_i \theta_k - P_k \theta_i = \frac{\partial}{\partial u_\alpha}\left[\Phi_{\alpha\beta}\left(\frac{\partial\theta_i}{\partial u_\beta}\theta_k - \frac{\partial\theta_k}{\partial u_\beta}\theta_i\right)\right]$$

schreibt und dann (38), (68), (53) heranzieht.

Vermöge (89), (123), (126), (127) erscheint die Grösse $\delta^2 F$ als eine Summe von sieben Invarianten des Gewichtes 1. Demgemäss *zerlegt sich die zweite Variation*

$$\delta^2 I = \int^{(n)} \delta^2 F\, du$$

in sieben parameterinvariante Integrale

$$\left.\begin{aligned} \delta^2 I = & \int^{(n)} \delta t\, du + \int^{(n)} W\delta(\theta\omega)\, du - \frac{1}{2}\int^{(n)} \triangle(\omega^2)\, \theta du \\ & + \int^{(n)} \triangle_1(\omega)\, \theta du + \int^{(n)} \omega^2 \theta^{-1} [\triangle_1(\theta) - \sum_i \triangle_1(\theta_i)]\, du \\ & + \int^{(n)} \omega \frac{\partial W}{\partial x_i} \delta x_i \theta\, du + \int^{(n)} \omega\theta^{-1}\theta_k (P_i\theta_k - P_k\theta_i)\, \delta x_i\, du. \end{aligned}\right\} \quad (128)$$

Bei festgehaltener $(n-1)$-stufiger Begrenzung verschwinden das erste und dritte von ihnen.

Dass unter dieser Annahme die Formel (128) für eine Extremale $W = 0$ in diejenige übergeht, welche ich in diesem Sonderfalle a.a. O.[6]), (69) angegeben habe, bestätigt man so: Weil dann nach [6]), S. 139

$$\frac{\partial W_k}{\partial x_i} + P_i\, \theta_k = \frac{\partial W_i}{\partial x_k} + P_k\, \theta_i$$

ist, vereinfacht sich die Summe des sechsten und siebenten Integranden nach [1]), (8) und nach (42), (124) wie folgt:

$$\omega\theta^{-1}\theta_k\left(\frac{\partial W_k}{\partial x_i} + P_i\theta_k - P_k\theta_i\right)\delta x_i = \omega\theta^{-1}\,\theta_k \frac{\partial W_i}{\partial x_k}\delta x_i = \omega^2 \frac{\partial W_k}{\partial x_k}.$$

Physics. — *The best method of measurement of a resistance thermometer.* (21st Communication of results obtained by the aid of the "VAN DER WAALS-Fund"). By A. MICHELS and P. GEELS. (Communicated by Prof. J. D. VAN DER WAALS Jr.).

(Communicated at the meeting of December 17, 1927).

The present communication forms a continuation of the 19th communication [1]) in which the replacement of the ice-point of the thermometer scale by another fixed point, reproducible to within $^1/_{4000}{}^{\circ}$, was proposed. In connection with the desired accuracy, it was found necessary to investigate the factors, which determine the accuracy of a resistance thermometer, and to find how the influence of these factors could be reduced to a minimum.

The following considerations are also partially applicable to other observations. The use of a resistance thermometer depends on the change of the resistance of a measuring wire with temperature, and other influences which result in an alteration to the resistance (for example the pressure effect) are amenable to similar treatment.

Besides the external factors, such as the choice of galvanometer, the accuracy of the resistance boxes used etc., which influence any resistance measurement, the most troublesome factor in the use of a resistance thermometer is the temperature rise of the measuring wire resulting from the measuring current.

In an absolute temperature measurement it is therefore desirable not to work with current which results in a temperature rise of the wire greater than the accuracy with which it is desired to establish the temperature. [2])

The temperature rise is determined by two factors, the amount of heat evolved by the Joule effect and the velocity with which this heat is dissipated to the surroundings. The latter is very greatly influenced by the construction of the thermometer and the best construction will be that with which the heat is dissipated as rapidly as possible, in other words a thermometer with as small a lag as possible, a factor also very desirable for other reasons.

The lower limit of this lag will be largely determined by other conditions such as insulation, stability etc. which will not be considered further.

[1]) These Proceedings, **30**, p. 1017 (1927).

[2]) N.B. Actually it should be permissible to work with a constant temperature rise, but it would then be necessary to be certain of the constancy.

Even though it is understood that the best arrangement has been chosen so far as these factors are concerned, a large variation may still be made as to the length and diameter of the wire and the measuring current.

The temperature of the wire is given by the expression:

$$\delta t = \beta \frac{i^2}{d^3}$$

where β is a constant, i the measuring current and d the diameter of the wire. This relation between δt and i has already been tested and established [1]). The temperature rise of the wire is therefore proportional to the square of the measuring current.

From the 20[th] communication (to be published in the following number of these Proceedings, vol. **31**) it follows that, if i is the current strength in the measuring wire and dR an arbitrary alteration to the resistance R, then the galvanometer deflection is given by

$$a = C \frac{i \dfrac{dR}{R} \times R}{\sqrt{g + R_0 + R}},$$

when a moving coil galvanometer within its aperiodic limits is used.

If a moving coil galvanometer in a constant field is used, this expression only differs in the numerator, the root being replaced by the first power.

As a moving coil galvanometer is usually used, the derivation will be given for this instrument in its aperiodic limit and only the result given for the other case, which may be obtained in exactly the same way.

It is hardly necessary to mention that the formulae used hold for almost any circuit, whether a potentiometer, a differential or a bridge method is used. In the last circuit it is understood that $n \gg 1$ (loc. cit. for the notation used). These conditions are not sufficient in the case of the THOMSON Bridge, but this bridge is of little importance for the present purpose. The only alteration that can occur is in the value of R_0, which disappears in some cases, and which is always small compared to the galvanometer resistance g (loc. cit.).

In order to simplify the present calculations $g + R_0$ has been replaced by G, so that the above expression becomes

$$a = C \frac{i \dfrac{dR}{R} \times R}{\sqrt{G + R}}.$$

It is at once apparent from this expression that the accuracy of the measurement is directly proportional to i, whilst, as already observed, δt is proportional to i^2. These are therefore two opposed influences.

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[1]) 17th Communication of the VAN DER WAALS Fund. These Proceedings **30**, p. 47.

The limit of the measurable temperature interval is that interval which is just equal to the temperature rise of the wire itself.

It is thus necessary to know the length and diameter of the wire for a given resistance R, with which a minimum temperature rise is obtained.

As already indicated

$$\delta t = \beta \frac{i^2}{d^3}$$

whilst the resistance R is given by

$$R = \gamma \frac{l}{d^2}$$

where γ is a constant and 1 the length of the wire.

From these expressions it follows that the smallest temperature rise with a given current is obtained when d is as large as possible, and, therefore, for a given R, when the wire is as long as possible.

This is also clearly shown by eliminating d from the above two expressions to give

$$\delta t = A i^2 \left(\frac{R}{l}\right)^{3/2}$$

(A is a constant).

Thus, from either point of view, it is desirable to make the wire as long as possible. Other external factors, such as the winding space, necessary distance for insulation etc., will determine the value of l. If it is assumed that l is made as large as possible in relation to the method of measurement, l may be considered as a constant and will disappear as a variable from the equations.

The problem may then be solved as follows.

Let Δt be the temperature alteration which it is desired to measure and δt the temperature increase which may be tolerated (this may be left undecided, if a relation is afterwards established between Δt and δt).

As δt has been chosen, it may be treated as a constant.

The deflection of the galvanometer is given by

$$\alpha = C \frac{i \frac{dR}{R} R}{\sqrt{G+R}}$$

in which $\frac{dR}{R}$ is proportional to Δt.

Put $C \frac{dR}{R} = D \Delta t$

$$\alpha = D \Delta t \frac{iR}{\sqrt{G+R}}$$

It is now necessary to find the minimum value of Δt under the given

condition that the temperature rise is not greater than δt. This is a limiting condition capable of mathematical determination

$$\delta t = \beta \frac{i^2}{d^3} \quad \text{whilst} \quad R = \gamma \frac{l}{d^2}.$$

Eliminating d and bringing all the constants (including δt) under one letter, the limiting condition may be expressed as

$$i^4 R^3 = E.$$

There is an experimental value of α, which is the smallest value observable. Let this be μ, then the smallest value of $\triangle t$ is given by

$$\mu = D \triangle t \frac{iR}{\sqrt{G+R}}$$

when $\frac{iR}{\sqrt{G+R}}$, which may be represented by z, is made as large as possible within the limiting conditions.

If z is plotted in a space diagram as a function of i and R (the x- and y-axis respectively), the question is reduced to the determination of the maximum of a surface with a border condition. This condition defines a space curve on the surface. From the expression $z = \frac{iR}{\sqrt{G+R}}$ it appears that the surface is regular and that the boundary lines go through the R-axis and run parallel to the $(z - i)$ surface. The surface therefore possesses no *absolute* maximum, although it reaches a maximum value on the boundary, and the question is therefore reduced to the determination of the maximum of the space curve defined by the two equations

$$z = \frac{iR}{\sqrt{G+R}}$$

$$i^4 R^3 = E.$$

This determination is made as follows:

$$F = K \frac{iR}{\sqrt{G+R}} + \lambda (i^4 R^3 - E) \qquad i^4 R^3 - E = 0$$

$$F'_{(i)} = K \frac{R}{\sqrt{G+R}} + 4 \lambda i^3 R^3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$F'_{(R)} = K \frac{i}{\sqrt{G+R}} - \frac{1}{2} K \frac{iR}{\sqrt{G+R}^3} + 3 \lambda i^4 R^2 = 0 \quad . \quad . \quad (2)$$

(1) and (2) give:

$$\frac{K}{\sqrt{G+R}}+4\lambda i^3 R^2=0. \quad \ldots\ldots \quad (3)$$

$$\frac{K}{\sqrt{G+R}}-\frac{1}{2}\frac{KR}{\sqrt{G+R}^3}+3\lambda i^3 R^2=0.$$

$$\overline{\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad}$$

$$1/2\frac{KR}{\sqrt{G+R}^3}+\lambda i^3 R^2=0$$

substituting in 3

$$\frac{K}{\sqrt{G+R}}-2\frac{KR}{\sqrt{G+R}^3}=0$$

$$R=G.$$

The result $R=G$ is independent of the value of E and therefore of δt and hence holds for the case chosen $\delta t=\triangle t$.

In the latter case it is possible to obtain a simpler solution, using the same proof that the maximum lies on the border curve, as the expression $\triangle t=\beta\frac{i^3}{a^3}$ is not then a condition for the maximum, but is an absolute equation.

Eliminating d between this equation and $R=\gamma\frac{l}{d^2}$ gives

$$\triangle t^2=H i^4 R^3.$$

Solving for i and substituting the value in

$$a=D\triangle t\frac{iR}{\sqrt{G+R}}.$$

$$=L(\triangle t)^{3/2}\frac{R^{1/4}}{\sqrt{G+R}} \qquad (H \text{ and } L \text{ constant})$$

or for minimum $a=\mu$

$$\mu=L(\triangle t)^{3/2}.\frac{1}{\sqrt[4]{\frac{(R+G)^2}{R}}}.$$

The smallest value of $\triangle t$ is obtained when $\frac{(G+R)^2}{R}$ is a minimum.

Differentiation gives $G=R$.

It thus appears that the best value is found when R is made equal to G. R is therefore determined, and where l is fixed, d is also established. A simple numerical calculation shows that the maximum is not very pronounced and that a variation of 50 % in d does not make any appreciable alteration to the best conditions.

The value of $\triangle t$ corresponding to the value of R can only be calculated when the necessary experimental data relating to the radiation, galvanometer sensitivity etc. are known.

The above deduction is only practicable when the value $R = G$ lies within the limits in which the variable shunt resistance can be regulated.

A similar calculation for a moving coil galvanometer in a constant field gives $R = {}^1/_3\, G$.

In conclusion a few notes on a circuit with overlapping shunts will be given in connection with the above.

In this circuit it is only the difference between the two currents passing through the galvanometer circuits, that acts as a directing current on the galvanometer. This results in a large current being passed through each of the galvanometer coils and the limitations of the galvanometer current being reached before those of the current in the measuring wire. This inconvenience may be avoided in the following way:

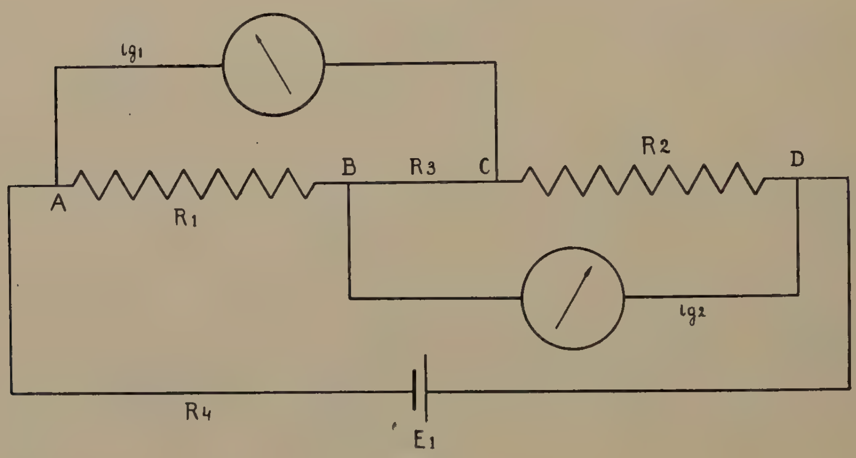

Fig. 1.

Fig. 1 is a schematic diagram of a Kohlrausch circuit, the current commutation being omitted.

Take, as is sufficient for the present derivation, the case when the two resistances R_1 and R_2 are equal. The galvanometer is adjusted to give no deflection. An alteration of the resistance from $R \rightarrow R + dR$ gives a deflection, which is determined by the algebraic sum of i^{g_1} and i_{g_2}.

In the calculation of the deflection dR may be replaced by an $E.M.F.$ idR, E left out of consideration and R_4 broken.

In the equilibrium condition an $E.M.F.$ in R_3 will also make no alteration to the algebraic sum of i_{g_1} and i_{g_2}.

According to the superposition law an EMF may be introduced into R_3 as well as into R_4. The two EMF's together will not influence the algebraic sum $i_{g_1} + i_{g_2}$. The latter, and therefore the galvanometer deflection, will remain exclusively determined by idR. If E_1 and E_2 are

chosen opposite in sign i_{g_1} and i_{g2} may be both reduced to a very small value by the exact choice of E_1 and E_2.

The Steinwehr commutator must be modified to incorporate this addition.

It is desirable to place a shunt across both E_1 or E_2 in order to obtain an exact regulation.

Figure 2 gives the potential fall in the main circuit.

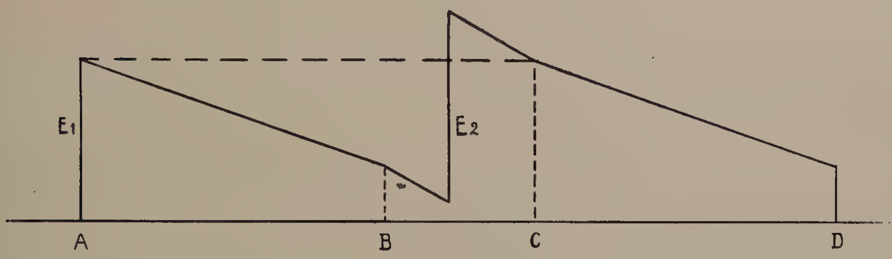

Fig. 2.

Geology. — *Alkaline rocks of the volcano Merapi (Java) and the origin of these rocks.* By H. A. BROUWER.

(Communicated at the meeting of February 25, 1928).

Since HARKER [1]) and BECKE [2]) have separated an atlantic from a pacific facies of eruptive rocks, of wich the atlantic rocks with the same acidity are richer in alkalies, many exceptions to this rule have been found. The opinion that alkaline rocks are connected with radial and sub-alkaline rocks with tangential movements of the earth-crust has been carried very far, but it has been stated, that in different cases there may be but a remote connection with tectonic movements as f.i. by the influence of these movements on the erosion, by which products of differentiation formed at various depths, can be found now at different places at the surface.

Of late different opinions on the origin of alkaline rocks have been put forward, partly emphasizing the absorption of limestone by sub-alkaline magmas (DALY) [3]) partly emphasizing differentiation during crystallization (BOWEN) [4]).

Nearly all the volcanoes of Java have produced pyroxene andesites and basalts, near the north coast of Java (Moeriah, Loeroes, Ringgit, island of Bawean) however, there are volcanoes, from which leucite- and nepheline-bearing rocks have been erupted.

For the study of the differentiation in a volcanic magma, which is not accessible for direct observation, the study of the xenoliths in the volcanic rocks is of much importance. If reaction with limestone really produces alkaline magma the study of exomorphic and endomorphic contactmetamorphism of xenoliths could give a decisive answer. Till now my studies on contactmetamorphism in the East-Indies did not give any indication on the production of alkaline magma in connection with xenoliths of limestone. The minerals formed under the influence of the magma in the limestone are

1) A. HARKER. The Natural History of Igneous Rocks. Article in Science Progress VI, 1896, p. 12 and in bookform 1909.

2) F. BECKE. Die Eruptivgebiete des böhmischen Mittelgebirges und der amerikanischen Andes. Atlantische und Pazifische Sippe der Eruptivgesteine. Tscherm. Min. Petr. Mitt. XXII, 1903, p. 209—265.

3) R. A. DALY. Origin of alkaline rocks. Bull. Geol. Soc. America Vol. **21**. 1910. p. 87. Ibid. Igneous Rocks and their origin. 1914. blz. 410. Ibid. Genesis of alkaline rocks. Journ. of Geol. XXVI, 1918. p. 97.

4) N. L. BOWEN. The later stages of the evolution in igneous rocks. Journ. of Geol. XXIII, suppl. N^0. 8. 1915. Ibid. Crystallization-Differentiation in igneous magmas. Journ. of Geol. XXVII, 1919, blz. 393. Ibid. The behaviour of inclusions in igneous magmas. Journ. of Geol. XXX, suppl. p. 513.

the same as those in other volcanic regions; they are in the first place lime-bearing minerals, as wollastonite, pyroxene, idocrase, garnet, anortite. By the influence of volatile constituents of the magma several other minerals are also formed as for instance in the metamorphosed xenoliths of limestone in Middle- and South-Italy, of which the Somma is a well-known locality.

Rocks and Xenoliths of the Merapi.

The sediments of most of the volcanoes on the island of Java belong to the pyroxene andesites and basalts and the rocks of the Merapi are no exception on this rule. VERBEEK and FENNEMA [1]) mention different pyroxene andesites mostly with a small olivine content, pyroxene andesite with some amphibole is also found.

The top of the volcano is formed by a lava dome of pyroxene andesite in which amphibole and very little olivine is found [2]). At my request Dr. G. L. L. KEMMERLING collected a number of xenoliths in the volcanic rocks of the Merapi, which were sent to me by the "Dienst van den Mijnbouw" and this collection was completed by myself during an ascension of the volcano in October 1923. Most of these xenoliths have been collected in rocks of the lava dome, some samples of the lava dome were studied under the microscope, they are pyroxene andesites with hyperstene in a much smaller quantity and in smaller crystals than augite. The numerous plagioclase phenocrysts have a zonal structure with frequent alternations of more basic and more acid zones so that the margin is only slightly more acid than the bytownitic central part. Larger crystals of ore are also found. In some slides were found rests of brown amphibole, which are mostly strongly resorbed. Some large crystals of amphibole which are up to several centimeters in length can more likely be considered as xenoliths than as real elements of the volcanic rocks.

The groundmass consists of plagioclase, pyroxene, iron ore and a varying quantity of glass.

Of the xenoliths we only mention those of sedimentary origin. They are principally metamorphic limestones, sandstones and arkoses. Only the metamorphic limestones are of importance for our present subject.

Metamorphic Limestones.

There are different mineral associations belonging to the metamorphic limestones, which appear partly in the same xenoliths but are also found separately as different xenoliths. Calcite is found in several xenoliths,

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1) R. D. M. VERBEEK en R. FENNEMA. Description géologique de Java et Madoera I. p. 322.

2) G. L. L. KEMMERLING. De hernieuwde werking van den vulkaan G. Merapi (Midden-Java) van begin Augustus 1920 tot en met einde Februari 1921. Vulkanologische mededeelingen. N⁰. 3. 1921. p. 28.

Fig. 1. Trachyte with porous groundmass with much glass and numerous small lath-shaped crystals of orthoclase. Enlarged × 40.

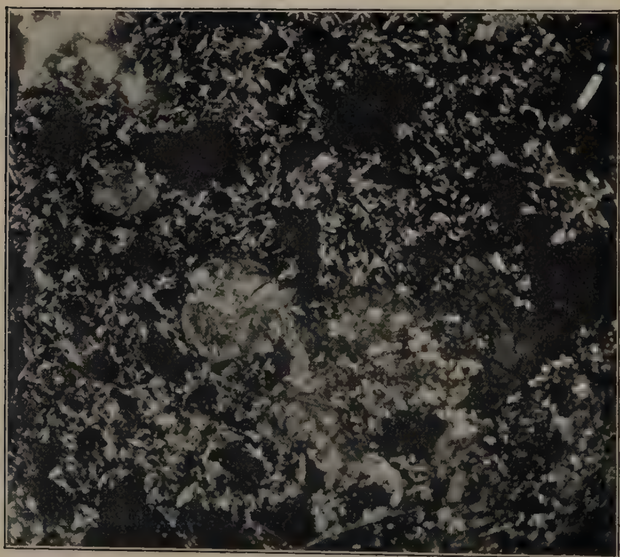

Fig. 2. Leucite phonolite with phenocrysts of leucite in a groundmass with much glass and lath-shaped crystals of orthoclase. In the fig. four leucite phenocrysts with strong optic anomalies are visible. Enlarged × 40.

sometimes in very small quantities only. The following mineral associations can be distinguished.

1. *Wollastonite-diopside.* They are partly fine-grained mixtures of these minerals with small quantities of ore, plagioclase and carbonate. More coarsely grained parts in these xenoliths consist of wollastonite and diopside, in which small pyroxene crystals often are enclosed by larger wollastonite crystals. At the contact of the andesite and the xenoliths a narrow zone of a substance rich in iron ore is found.

In other xenoliths parts with much wollastonite alternate with green parts with much pyroxene. Along the contact of andesite and xenoliths is found a zone, which is rich in pyroxene and iron ore with plagioclase and brownish glass; a light-brown glass is also found between and in the crystals of wollastonite.

2. *Garnet-wollastonite-epidote.* They are xenoliths with much carbonate. Yellowish-green garnet and a mineral of the epidote group with a low double-refraction are found in a fine-grained mixture. Wollastonite is principally found outside this mixture and is also intergrown with carbonate.

3. *Garnet-wollastonite-epidote-plagioclase-diopside.* Parts of the xenolith consist of a mixture of strongly double-refracting epidote and light-brownish yellow garnet, which sometimes enclose parts which nearly entirely consist of wollastonite. Other parts consist of a plagioclase-pyroxene mixture. The xenolith forms the central part of a homoeogeneous xenolith consisting of a fine-grained mixture of plagioclase and diopside which at the margin of the metamorphic limestone is characterized by a strong increase of the pyroxene content with regard to the plagioclase. The xenolith has a rather considerable carbonate content.

4. *Contactmetamorphic limestone with zonal structure.* A block of large dimensions ($50 \times 50 \times 30$ cm) collected on the lahar field of the Kali Batang has a zonal structure and has been described by KEMMERLING as a greyish-green schist, probably belonging to the diabase- and chlorite schists, which are found at the surface among the pre-Tertiary rocks of the Djiwo mountains South of Klaten (KEMMERLING loc. cit. p. 29). It is however a contactmetamorphic limestone, in which zones or lenses of different mineralogical composition and colour alternate.

The mineral associations in a certain zone often change from place to place, in different zones certain minerals are predominating, which can however appear also in smaller quantities in other zones. There are for instance light-coloured zones, which consist of a mixture of wollastonite and carbonate in varying quantities, whether or no with plagioclase and augite, which also can predominate together. Leucite is a common mineral which is found in different zones, also with basic plagioclase, biotite and augite. Some zones consist of orthoclase and augite with some calcite. The leucite-bearing zones represent phanerites with preponderant leucite, which are extremely scarce among the igneous rocks.

Of great importance are phonolitic and trachytic zones mostly with phenocrysts of leucite or orthoclase and with much glass in the groundmass. These will be described in some more detail.

Trachytes and Phonolites.

As has been mentioned above these are found as zones of varying composition in the contactmetamorphic limestones with zonal texture. Two main types can be distinguished :

a. phenocrysts of orthoclase, sometimes with the fissures of sanidine, are imbedded in a groundmass with much glass and numerous small lath-shaped crystals of orthoclase or acid plagioclase. Pyroxene microlites also occur, they partly have a small extinction-angle and are optical negative which indicates the presence of the aegirine molecule (fig. 1).

b. The groundmass differs from that mentioned sub *a* by the abundance of leucite in more or less idomorphic or rounded crystals, leucite is also found as small phenocrysts or is restricted to the phenocrysts (fig. 2). The optic anomalies, sometimes with distinct polysynthetic twins, a strong potash reaction and the positive double-refraction confirm the determination as leucite.

In both types (*a* and *b*) calcite, apparently of magmatic origin, is locally found. Both types form transitions into mineral associations, in which the glass-bearing groundmass disappears. These partly leucite-bearing phanerites have been mentioned already above. Besides, orthoclase is not the only mineral, which appears as phenocrysts, for exceptionally a rather basic plagioclase is also found, which shows the exsistence of transitions to more andesitic types. Also augite is locally found as small phenocrysts in the zones mentioned sub *a* and *b*. The great importance however lies in the presence of zones with the composition of trachytes and leucite phonolites, while for the rest the magma of the Merapi has produced pyroxene andesites only.

A connection between the origin of the trachytic and phonolitic zones and a reaction of the pyroxene andestic magma with the metamorphic limestones, in and at the margin of which the zones are found, is abvious. The peculiar texture and the great dimensions of the block of metamorphic limestone might be indications that it is a part of the wall-rock, which had already been metamorphosed, before it was detached and enclosed in the magma. The great importance of the study of the xenoliths lies in the conclusions that can be obtained with regard to the processes, that take place in the deeper parts of a volcano. There are no experimental results available which illustrate the assimilation of limestone by an andesitic or basaltic magma, while the batholitic assimilation can only be judged by its consequences, which are explained in different ways. The phenomena, which have been described above have not exactly the same value as an experiment because the supposition is possible that a concentration of the volatile constituents with the alkalies took place in the conduit

and that the association with limestone took place afterwards. If however such a differentiation by crystallization produced alkaline rocks in the Merapi, the expectation would be founded, that from the Merapi and from the numerous other volcanoes of Java, which produced pyroxene andesites and basalts only, at least a single piece of alkaline rock, without connection with limestone, would have become known. These expectations do not agree with the facts.

Distribution of alkaline rocks on Java.

In how far assimilation of limestone or differentiation by crystallization can be considered to be the cause of the origin of the alkaline magmas must be considered in every separate case. And though BOWEN states that assimilation has been but a small factor in the production of the great variety of eruptive rocks, be does not exclude the possibility of limestone-assimilation f.i. for the formation of melilite basalt and some other alkaline rocks.

Although this question cannot be decided for the volcanoes of Java, it is of importance to consider in how far it is possible, that assimilation of limestone has played a part in the production of the alkaline rocks.

Alkaline rocks are only known near the north coast of the eastern part of Java (Moeriah near Semarang, island of Bawean north of Soerabaya, Loeroes west of and Ringgit east of Besoeki). The division between this region, where alkaline rocks are found and by far the greatest part of Java where the volcanoes erupted pyroxene andesites and basalts only, cannot be made too sharply for besides alkaline rocks subalkaline rocks are found among the products of the same volcanoes. Leucite basalt, amphibole andesite, basalt (partly orthoclase-bearing) and trachyandesite are known from the Loeroes, and from the Ringgit we know leucitite, leucite basalt, tephrite and basanite and also trachyandesite without leucite, andesite and basalt. On the other hand the Merapi, a typical representative of the Javanese andesite- and basalt volcanoes has produced trachyte and phonolite although in a small quantity. As a matter of course our knowledge of the substratum of the volcanoes is incomplete, but a facies rich in limestones is characteristic for the Tertiary near the north coast of Rembang and in Madoera, while thick beds of limestone which must have given a great possibility for assimilation, are found near alkaline rocks in South-Celebes f.i. in the vicinity of the Peak of Maros. Perhaps the dip of the limestones of the Gg. Toegoe, south-east of Klaten can be connected with the occurrence of xenoliths of limestone in the products of the Merapi. The various explanations which have been given for the origin of alkaline magmas show certain features in common despite great differences of emphasis, as has been stated already by SMYTH [1]), who considers the origin

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[1]) C. H. SMYTH. The chemical composition of the alkaline rocks and its significance as to their origin. Amer. Journ. of Science. XXXVI, 1913, blz. 33.

of alkaline rocks to be principally affected through the agency of mineralizers, the influence of which is also taken into consideration by other authors.

The pneumatolytic phenomena, which take place in the magma during the long periods of dormancy of a volcano could favour the production of an alkaline magma on a larger scale than we described for the volcano Merapi. An important addition of lime will not be prevented by the progressive crystallization of the magma if the magma is very fluid and rich in fugitive constituents. Without reaction with limestone the alkalies could also be concentrated but there is not one of the numerous volcanoes of Java, which produced pyroxene andesites and basalts only, where we can find an indication, that alkaline differentiates have been produced in this way.

Lastly also the mechanical hypothesis to explain the distribution of alkaline rocks cannot be left entirely out of consideration. For Java the difference in stability between north- and southcoast is a striking feature, but there are no data from which could be concluded that the differentiation is influenced by the doubtless strongly changing crustal movements.

A further study of the limestone-xenoliths in the products of the different volcanoes of Java will be of interest. In the first place the attention can be drawn to those volcanoes near the north coast, of which no alkaline rocks are known.

Physics. — *Experiments on the velocity distribution in the boundary layer along a rough surface; determination of the resistance experienced by this surface.* By B. G. VAN DER HEGGE ZIJNEN. (Mededeeling N⁰. 10 uit het Laboratorium voor Aerodynamica en Hydrodynamica der Technische Hoogeschool te Delft.) (Communicated by Prof. J. D. VAN DER WAALS Jr.)

(Communicated at the meeting of February 25, 1928).

§ 1. *Introduction.*

In a discussion of the experimental data concerning the resistance experienced by the flow through pipes and channels with rough walls, HOPF [1]) deduced that this resistance is proportional to the square of the mean velocity, at least if the state of motion is entirely turbulent. According to FROMM [2]) the same law holds for the pressure drop in a rectangular channel, the walls of which are composed of surfaces having a well defined roughness. Evidently in these cases the resistance coefficient is independent of REYNOLDS' number and is determined entirely by the dimensions of the channel and of the projections on its surface.

However, the numerous experiments on the resistance of flow still need completion and extension; researches on the distribution of the velocity in the boundary layer, on the development of the boundary layer along the surface and more detailed experiments concerning the resistance of rough surfaces are of importance for a better understanding of the phenomena observed by several experimenters.

The opportunity for carrying out such researches in the Laboratory for Aerodynamics and Hydrodynamics of the Technical University at Delft presented itself in the beginning of 1925, when by the courtesy of Prof. VON KÁRMÁN a sheat of "waffle-plate", of about the same aspect as used by FROMM in his researches on rectangular channels, was put at our disposal. The experiments were finished in 1927, when the measurements on the velocity distribution could be completed and checked by weighing the resistance experienced by a board, covered on both sides with waffle-plate, directly on a balance.

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1) L. HOPF. Abhandl. a. d. Aerodynamischen Institut der Techn. Hochschule Aachen III, 1924, p. 1; Zeitschr. f. angewandte Math. u. Mech. 3, 1923, p. 329.

2) K. FROMM. Abhandl. a. d. Aerodynamischen Institut der Techn. Hochschule Aachen III, 1924; Zeitschr. f. angewandte Math. u. Mech., 3, 1923, p. 339.

§ 2. *Summary of the principal theoretical data about the motion in the boundary layer.*

Before describing the experiments performed and discussing their results, some formulae concerning the flow in the boundary layer along a rough surface may be deduced.

The following considerations are based upon the hypothesis of Prof. VON KÁRMÁN [3]) that the resistance of a rough surface is entirely due to the head resistance of the projections. It might be expected that as this head resistance, at least above a certain value of REYNOLDS' number, follows the quadratic law, the same will be the case with the resistance experienced by the entire surface; this supposition is supported by the work of HOPF and FROMM mentioned above.

The velocity distribution in a section of the boundary layer will now be supposed to satisfy a relation of the form:

$$u = V\left(\frac{y}{\delta}\right)^n \quad \ldots \ldots \ldots \ldots \quad (1)$$

where u = velocity (parallel to the surface) in an arbitrary point of the section considered; V = velocity of the undisturbed flow just outside of the boundary layer; y = distance of the point in question from the surface; and δ = thickness of this section of the boundary layer.

This relation has to be verified by experiment. If the surface is not too rough, and if REYNOLDS' number for the boundary layer $\left(R_\delta = \frac{V\delta}{\nu}\right)$ is sufficiently high, it may be expected from the results of other researches that form. (1) will hold.

Although about the flow in the vicinity of the projections nothing can be predicted with certainty, following VON KÁRMÁN the velocity at the top of a projection of height h may be expressed by:

$$u_t = aV\left(\frac{h}{\delta}\right)^n \quad \ldots \ldots \ldots \ldots \quad (2)$$

where a is an unknown numerical constant.

The resistance of a projection will now be proportional to u_t^2, and the resistance experienced by the surface per unit area may be written:

$$\tau_0 = c\varrho V^2\left(\frac{h}{\delta}\right)^{2n} \quad \ldots \ldots \ldots \ldots \quad (3)$$

In order to deduce a relation between δ and the distance x of the section

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3) TH. VON KÁRMÁN. Ueber die Oberflächenreibung von Flüssigkeiten, Vorträge aus dem Gebiete der Hydro- und Aerodynamik, Innsbruck 1922 (Berlin 1924), p. 146.

considered from the leading edge of the surface, we may use the formula given by VON KÁRMÁN [4]) for the loss of momentum. It has to be born in mind, however, that the experiments described below relate to a surface mounted in a wind tunnel, in which case the velocity V outside of the boundary layer is not constant, but increases down stream in consequence of the narrowing of the passage of the flow, caused by the growth of the boundary layers along the tunnel walls and along the surface.

The increase of V is rather small; putting:

$$\frac{dV}{dx} = \beta V \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

we may neglect terms as $\beta^2 x^2$, $\beta^3 x^3, \ldots$, in the following formulae, even when x is equal to the length l of the plate.

In this case the integral of (4) is:

$$V = V_0 (1 + \beta x) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4^a)$$

The loss of momentum in the boundary layer per unit length will be:

$$\frac{d}{dx}\int_0^\delta u^2\, dy - V\frac{d}{dx}\int_0^\delta u\, dy = -\frac{\tau_0}{\varrho} + \delta V \frac{dV}{dx} \quad . \quad . \quad . \quad . \quad (5)$$

Inserting form. (1), (3) and (4) and dividing by V^2, equation (5) is reduced to [5]):

$$\frac{n}{(2n+1)(n+1)}\frac{d\delta}{dx} + \frac{2n^2+3n}{(2n+1)(n+1)}\beta\delta = c\left(\frac{h}{\delta}\right)^{2n},$$

the integral of which is, when $\delta = 0$ for $x = 0$:

$$\delta = \left\{\frac{(2n+1)^2(n+1)\, c}{n}\right\}^{\frac{1}{2n+1}} x^{\frac{1}{2n+1}}\, h^{\frac{2n}{2n+1}} \left\{1 - \frac{2n+3}{2}\beta x\right\} \quad . \quad . \quad (6)$$

Hence from form. (3), inserting the value of V from ($4a$) and that of δ from (6), we get:

$$\tau_0 = c\varrho V_0^2 \left\{\frac{n}{(2n+1)^2(n+1)\, c}\right\}^{\frac{2n}{2n+1}} \left(\frac{h}{x}\right)^{\frac{2n}{2n+1}} \{1 + (2n^2+3n+2)\beta x\} \quad . \quad (7)$$

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[4]) TH. VON KÁRMÁN. Ueber laminare und turbulente Reibung, Zeitschr. f. angewandte Math. u. Mech., **1**, 1921, p. 235 form. (5).

[5]) The relations for δ, τ_0 and c_f which are given here for the case of an accelerated flow, have been deduced by Prof. BURGERS.

The integral of (7) with respect to x, multiplied by the breadth b of the plate, gives the total resistance:

$$\left.\begin{aligned} W &= b\int_0^x \tau_0\, dx \\ &= c\varrho V_0^2\, b x^{\frac{1}{2n+1}} h^{\frac{2n}{2n+1}} \left(\frac{1}{2n+1}\right)^{\frac{2n-1}{2n+1}} \left(\frac{n}{(n+1)c}\right)^{\frac{2n}{2n+1}} \left\{1+\frac{2n^2+3n+2}{2n+2}\beta x\right\} \end{aligned}\right\} \quad (8)$$

This expression is simplified by introducing:

$$I = \varrho \int_0^\delta u\,(V-u)\, dy = \frac{n}{(2n+1)(n+1)}\varrho\, V^2 \delta \quad . \quad . \quad . \quad . \quad (9)$$

The quantity I will be called the defect of momentum in the boundary layer. It has to be noted that in (9) occurs the local value of the velocity, V, and not V_0.

By means of (4a), (6) and (9) form. (8) leads to:

$$W = bI\left\{1+\frac{(2n+1)^2}{2\,(n+1)}\beta x\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If n and c are found from experimental results for a known value of β, the behaviour of δ, τ_0 and the value of W in the case of an unlimited flow (where V is constant), are easily deduced by putting everywhere $\beta = 0$ in the formulae.

§ 3. *Experimental arrangements.*

In order to collect data about the distribution of the velocity in the neighbourhood of rough surfaces and to check the formulae deduced for δ and for the resistance, experiments were performed with two sheats of "waffle plate" of dural, as used f.i. for covering treadles. The first plate served only for the measurements in the boundary layer; later on, when a greater sheat of the metal was put at our disposal, a second plate with projections differing but slightly from those of the first, was used for measuring directly on a balance the total resistance experienced by a board covered on both sides with it and for determining the loss of momentum in the wake down stream. In order to compare the results of these experiments with those performed on the first plate, a few measurements of the velocity distribution in the boundary layer along the second plate were performed.

Waffle plate I. The surface may be characterized as follows: The projections had the shape of quadrilateral pyramids, arranged in regular horizontal and vertical rows without change; their mean height was

TABLE I. VELOCITY IN THE BOUNDARY LAYER (IN CM/SEC).

y cm.	WAFFLE PLATE I																																			WAFFLE PLATE II		
	$V=811$ cm/sec.														$V=1600$ cm/sec.							$V=2400$ cm/sec.							$V=3200$ cm/sec.							$V_p=2400$ cm sec.		
	$x=25$		50		75		100		125		150		175		25	50	75	100	125	150	175	25	50	75	100	125	150	175	25	50	75	100	125	150	175	198 cm.		
	t	v	t	v	t	v	t	v	t	v	t	v	t	v	v	v	v	v	v	v	v	v	v	v	v	v	v	v	v	v	v	v	v	v	v	t	v	y cm.
0.025	380	39	354	36	230	35	310	35	324	19	300	18	317	22	139	117	74	62	56	27	112	261	228	114	139	114	324	217	368	388	210	221	185	335	243	1018	158	0.025
0.050	428	90	384	81	325	90	350	74	377	58	361	31	375	66	256	208	176	132	92	109	201	420	370	285	262	188	458	370	594	590	416	385	346	511	392	1142	295	0.050
0.075	462	152	425	140	384	121	371	125	400	106	396	64	396	122	369	316	272	236	184	202	282	584	515	429	393	316	615	520	761	761	625	510	538	686	548	1211	445	0.075
0.100	476	236	439	214	409	227	400	196	425	176	416	116	425	185	484	405	384	345	295	292	421	745	655	566	532	455	772	676	955	949	820	701	778	773	730	1260	615	0.100
0.125	485	331	450	304	434	304	409	270	441	256	441	184	436	262	638	529	549	466	426	412	548	888	790	790	701	616	924	848	1185	1130	1060	911	961	973	956	1296	824	0.125
0.150	520	396	466	361	445	361	425	313	452	341	450	270	445	327	734	635	645	579	530	538	615	1056	912	950	851	771	1045	980	1386	1295	1320	1120	1204	1290	1160	1340	1011	0.150
0.175	529	434	485	400	459	389	445	342	461	379	459	334	461	374	790	692	721	656	630	625	681	1176	1025	1060	950	911	1130	1050	1586	1415	1460	1225	1384	1475	1340	1384	1116	0.175
0.200	540	450	490	412	485	416	450	374	471	409	471	379	470	404	853	751	780	691	691	697	761	1253	1135	1160	1020	1050	1185	1120	1650	1585	1595	1410	1544	1545	1450	1421	1190	0.200
0.250	573	493	515	450	490	450	476	412	489	432	484	415	485	434	900	818	800	725	780	773	810	1421	1232	1240	1090	1220	1240	1225	1864	1705	1740	1500	1665	1700	1630	1513	1310	0.250
0.300	592	525	529	471	511	459	485	445	501	459	494	441	501	450	999	870	830	809	821	818	841	1513	1310	1310	1175	1285	1332	1320	1975	1805	1850	1620	1798	1780	1730	1568	1384	0.300
0.400	621	573	557	498	529	493	511	484	520	489	525	476	520	490	1082	942	900	890	898	906	911	1681	1460	1440	1360	1430	1430	1415	2210	1935	1990	1820	1850	1936	1884	1648	1513	0.400
0.500	666	618	579	539	549	529	539	493	541	510	539	494	532	520	1163	1010	961	955	950	949	961	1823	1545	1550	1460	1520	1513	1500	2395	2095	2110	1950	1971	2061	1990	1714	1600	0.500
0.600	709	653	621	573	573	557	561	529	557	529	551	515	551	549	1246	1082	1024	989	988	992	1010	1936	1640	1620	1555	1580	1570	1582	2580	2200	2245	2015	2100	2135	2060	1772	1714	0.600
0.700	729	689	640	601	592	573	573	557	573	552	568	532	568	568	1332	1150	1056	1024	1015	1015	1040	2061	1722	1660	1620	1640	1632	1620	2721	2315	2315	2100	2172	2190	2125	1815	1772	0.700
0.800	759	713	665	621	621	590	592	573	586	568	581	557	573	573	1436	1176	1096	1050	1035	1045	1060	2190	1840	1760	1660	1680	1675	1665	2860	2430	2400	2150	2256	2245	2200	1849	1823	0.800
0.900	770	749	681	651	638	620	618	581	600	576	592	573	581	581	1460	1246	1130	1120	1070	1070	1080	2304	1918	1820	1740	1740	1725	1710	2962	2520	2460	2237	2315	2305	2237	1884	1884	0.900
1.00	789	762	709	666	658	627	621	610	620	592	621	592	592	592	1513	1325	1183	1175	1100	1090	1095	2333	1955	1900	1760	1770	1765	1770	3062	2600	2525	2375	2375	2375	2294	1936	1936	1.00
1.25	811	784	740	715	689	665	658	630	640	620	640	621	621	621	1600	1421	1262	1262	1200	1155	1175	2400	2162	2005	1900	1880	1835	1840	3200	2860	2745	2500	2525	2440	2440	2043	1989	1.25
1.50	811	806	762	762	715	709	681	666	666	645	665	651	640	640	1600	1475	1355	1320	1262	1230	1262	2400	2245	2200	1990	1980	1935	1930	3200	3025	2845	2610	2610	2580	2540	2116	2061	1.50
1.75	811	811	789	780	749	715	713	702	688	668	688	673	665	666	1600	1521	1450	1380	1320	1310	1320	2400	2315	2240	2120	2050	1995	2010	3200	3115	2950	2765	2740	2725	2663	2190	2116	1.75
2.00	811	811	806	806	780	759	729	729	715	709	713	698	689	689	1600	1560	1500	1450	1360	1330	1380	2400	2345	2300	2240	2110	2060	2080	3200	3180	3100	2875	2845	2800	2767	2247	2190	2.00
2.50	811	811	811	811	806	784	770	762	761	749	750	722	715	722	1600	1600	1550	1500	1515	1430	1450	2400	2372	2380	2300	2240	2190	2180	3200	3200	3140	3080	2950	2960	2920	2381	2285	2.50
3.00	811	811	811	811	811	811	806	784	790	780	770	761	762	762	1600	1600	1585	1570	1560	1515	1520	2400	2400	2390	2380	2360	2275	2290	3200	3200	3160	3140	3114	3105	3030	2430	2333	3.00
4.00	—	—	—	—	811	811	811	811	811	811	811	811	811	811	1600	1600	1600	1600	1600	1600	1570	2400	2400	2400	2400	2400	2360	2375	3200	3200	3200	3200	3200	3180	3160	2510	2500	4.00
5.00	—	—	—	—	811	811	811	811	811	811	811	811	811	811	1600	1600	1600	1600	1600	1600	1600	2400	2400	2400	2400	2400	2400	2400	3200	3200	3200	3200	3200	3200	3200	2560	2560	5.00
$X=$	200	200	175	175	175	175	175	175	200	200	200	200	200	200	200	200	175	175	200	200	200	200	200	175	175	200	200	200	200	200	175	175	200	200	200	129	129 cm.	
Bar.	768	767	768	766	770	768	773	773	769	769	765	766	766	765	758	756	764	765	771	761	765	758	757	758	765	770	760	768	755	758	758	766	773	762	763	767	767 mm.	
Temp.	18.4	17.3	17.8	17.8	16.1	17.4	18.6	17.5	18.0	19.6	18.0	18.4	18.4	17.1	19.7	19.1	18.8	15.4	18.6	19.1	17.0	18.8	17.7	18.0	19.2	16.8	18.6	18.6	19.2	19.5	19.9	18.9	19.7	20.1	19.7	13.1	14.3 °C.	

1.7 mm; their distance in longitudinal direction 6.5 mm, in transverse direction 6.33 mm. The tops of the projections, as well as the valleys at their root, were slightly rounded off.

Comparing this with the rough surfaces used by FROMM, our "waffle plate I" seems to approach FROMM's "Waffelblech II", although the height of the projections of the latter plate was less (height 0.858 mm, distance of the tops 6.62 mm).

This surface was nailed on a rectangular wooden board of 189.5 $\times$ 50 cm, the leading edge of which was sharpened over 15 cm. The dimensions of the metal sheat allowed to cover one side only; however, the metal was bent around the leading edge of the board and covered the back over 10 cm. Two wooden clamps were provided at the uncovered side to give the board a greater stiffness.

This board was mounted vertically in the wind tunnel at a distance of 175 to 225 cm from the honey comb at the entrance (this distance is called X in table I).

The arrangement for the determination of the velocity distribution in the boundary layer was the same as that used in the experiments on a glass plate, described before [6]). The measurements were carried out with a hot wire anemometer (length of wire 37 mm, diameter 0.05 mm, heated to about 600° C above the surrounding atmosphere), which was mounted in a screw micrometer with displacement perpendicular to the plate. The distance between the hot wire and the surface could be regulated to 0.01 mm. The screw micrometer was mounted at the outside of the tunnel on a strong iron frame, which carried at the same time four round bars, two of which supported the upper edge of the plate and the two others the lower edge.

The displacement of the hot wire in the direction of the flow was performed by shifting the micrometer along the iron frame, without altering the position of the plate.

The mean velocity of the flow in the tunnel, V, was determined with a Pitot-tube, connected to an alcohol micromanometer; the value of V was kept constant by the experimenter. This Pitot-tube was mounted 268 cm behind the honey comb at midheight of the tunnel and 60 cm from the vertical tunnel wall facing the experimenter; the velocity indicated by this Pitot-tube will be called V_p.

Waffle plate II. The surface of the second metal sheat, which is called further on "waffle plate II", consisted of quadrilateral pyramids, 1.5 mm high, at distances of 6.65 mm in the longitudinal direction and of 6.4 mm in the transverse direction; hence it differed slightly from the surface of plate I.

Two sheats of this metal were fixed to the sides of a wooden board, which was sharpened at the leading edge over 15 cm, while the upper- and

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[6]) B. G. VAN DER HEGGE ZIJNEN. Measurements of the velocity distribution in the boundary layer along a plane surface (thesis Delft, 1924).

the lower edges were sloped over 5 cm in order to get a sharp edge in the direction of the plate. The sheats were fixed as tightly as possible to the board by means of small nails; at the leading edge, where the wood was thin, the sheats were riveted. The dimensions of the covered board are: length of the metal sheat: 199.4 cm, height (measured along the curved sheat): 50 cm; thickness of the board: 2.4 cm.

The arrangement for the measurement of the resistance is represented in fig. 1. The board C is supported by two thin steel wires D, E (diameter 0.33 mm, length 290 cm) in the vertical plane of symmetry of the tunnel; they passed the top of the tunnel with sufficient clearance. At the lower edge of the board two pairs of steel wires F and G were provided in order to avoid transverse oscillations; they were stretched slightly, so that the plate was free to swing longitudinally.

From the leading edge a steel wire H led to a sensitive balance B outside of the tunnel, the slope of which was 30°.30′; its end was fixed to the tunnel wall at K and a certain tension was given to this wire. The measurement of the forces acting on the model was performed in such a way that the other arm of the balance B was loaded with a determined weight, after which the wind speed in the tunnel was regulated until B was in balance. The resistance experienced by the plate was afterwards deduced from the weighed forces by resolving them graphically. During the measurements the distance of the leading edge of the board from the honey comb (X) was 158 cm; the lower edge of the board was 17.1 cm above the bottom of the tunnel, while the distance between board and the front wall of the tunnel was 41.6 cm. The Pitot-tube A was fixed at 143 cm behind the honey comb, 20 cm above the bottom and 52.4 cm from the front wall. The velocity V_p indicated by it will practically be equal to V_0 at the beginning of the board ($X = 158$ cm).

The resistance measured on the balance will be affected by the resistance of the steel wires carrying the board, and on the other hand by the suction at the trailing edge. In order to arrive at the true surface friction, corrections have to be applied for both influences.

The wire resistances are of minor importance; it is sufficient to deduce them from their length and diameter by means of the diagram for the resistance coefficients of cylinders given by PRANDTL [7]).

The suction at the trailing edge of the model is determined according a method also given by PRANDTL [8]) by mounting a brass tube of 9 mm diameter, provided with 9 holes of 0.8 mm in the space left between both metal sheats at the rear. This tube was closed at the bottom, while the upper

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[7]) L. PRANDTL. Ergebnisse der Aerodynamischen Versuchsanstalt zu Göttingen II (München 1923), p. 24.

[8]) L. PRANDTL. Ergebnisse der Aerodynamischen Versuchsanstalt zu Göttingen I (München 1921), p. 120.

end, projecting above the board, was connected with an alcohol micromanometer; the holes were directed down stream. For various values of V the suction in this space was compared to the static pressure on Pitot-tube A, which was supposed to be equal with sufficient accuracy to the pressure at

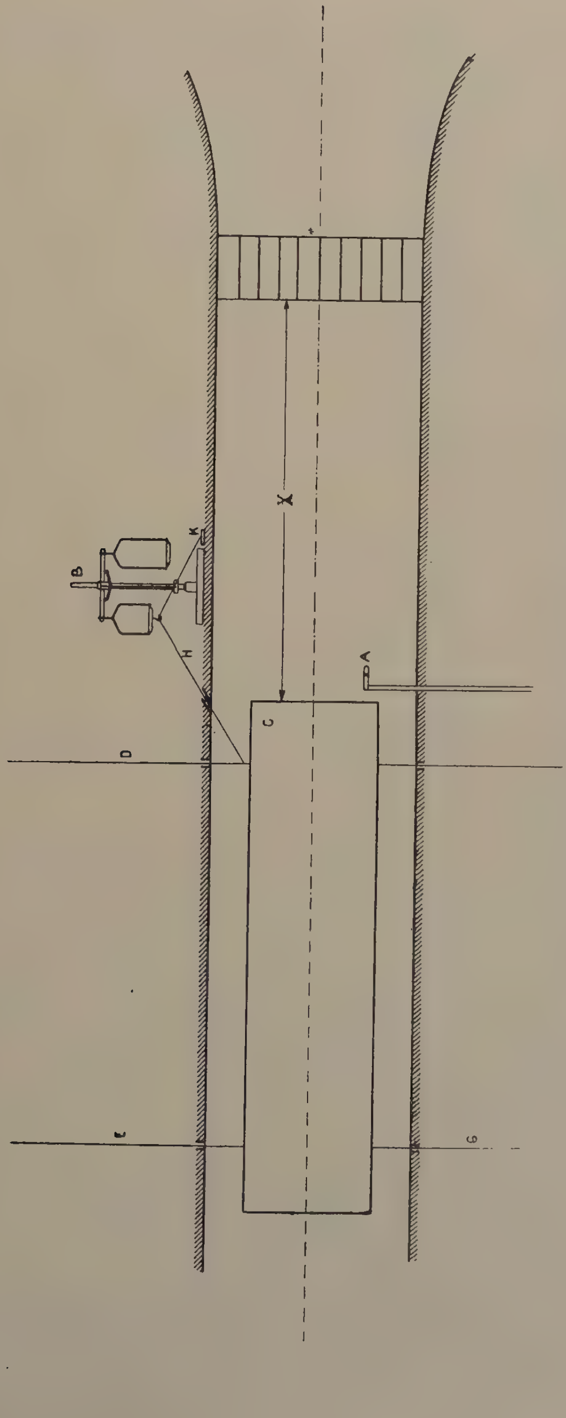

Fig. 1. Scheme of the arrangement for the direct determination of the resistance.

the leading edge of the plate. A second manometer was used for the determination of the air speed.

The pressure differences at the leading and at the trailing edge of the board, multiplied by the area of the trailing edge gives the suction experienced by the model.

This brass tube was absent during the resistance measurements with the balance.

The few measurements of the velocity distribution in the boundary layer along plate II were performed in the same way as those in the case of plate I, with the exception that here the first pair of iron bars, which held the board, was taken away and replaced by steel wires in order to avoid any disturbance of the flow as far as possible. The results of the velocity measurements with plate I had proved that these bars might cause a disturbance of the flow in the boundary layer in down stream sections. The pair of bars in the neighbourhood of the hot wire anemometer, however, could not be dispensed with, so that only the foremost pair, which gave the greater trouble, was removed.

§ 4. *Measurements in the boundary layer along plate I.*

The measurements on the velocity distribution along plate I were performed at a great number of distances from the leading edge and at various values of V.

As stated in § 2, the velocity outside of the boundary layer will increase down stream. However, during the experiments the wind speed was regulated in such a way that V had the same value at every value of x, as this made a check and a comparison of the results more easy.

As it proved to be impossible to determine the velocity gradient in the immediate neighbourhood of the surface in order to evaluate the shearing stresses, the velocity in the boundary layer was measured only at values of $y \geqslant 0.025$ cm. Some of the results have been collected in table I (the last columns of this table give the results of the measurements on plate II). Every velocity mentioned here is obtained as the mean of 6 readings. The indices t and v respectively relate to the measurements in which the distance y is estimated from a "top" or from a "valley" between the pyramids. The series t were observed immediately behind the series v and therefore the distance x had in this case to be increased by about 3 mm.

The experiments have been performed with $V = 811$ cm/sec in the sections $x = 5$, 10, 15, 20, 25, 37.5, 50, 62.5, 75, 87.5, 100, 125, 150 and 175 cm, y being reckoned in this case from a top and from a valley respectively; with $V = 1200$, 1600, 2000, 2400, 2800 and 3200 cm/sec the velocity was observed in the sections: $x = 25$, 50, 75, 100, 125, 150 and 175 cm for the series v only.

As might be expected, the series t and v show appreciable differences at the smaller values of y; they disappear for the greater part, however,

when in the series t the value of y is increased by the height h of the projections, $h = 1.7$ mm.

The results were plotted on a logarithmic scale (log u against log y); one of these diagrams, relating to the measurements at $x = 175$ cm, has been reproduced in fig. 2. By means of the diagrams it was inquired

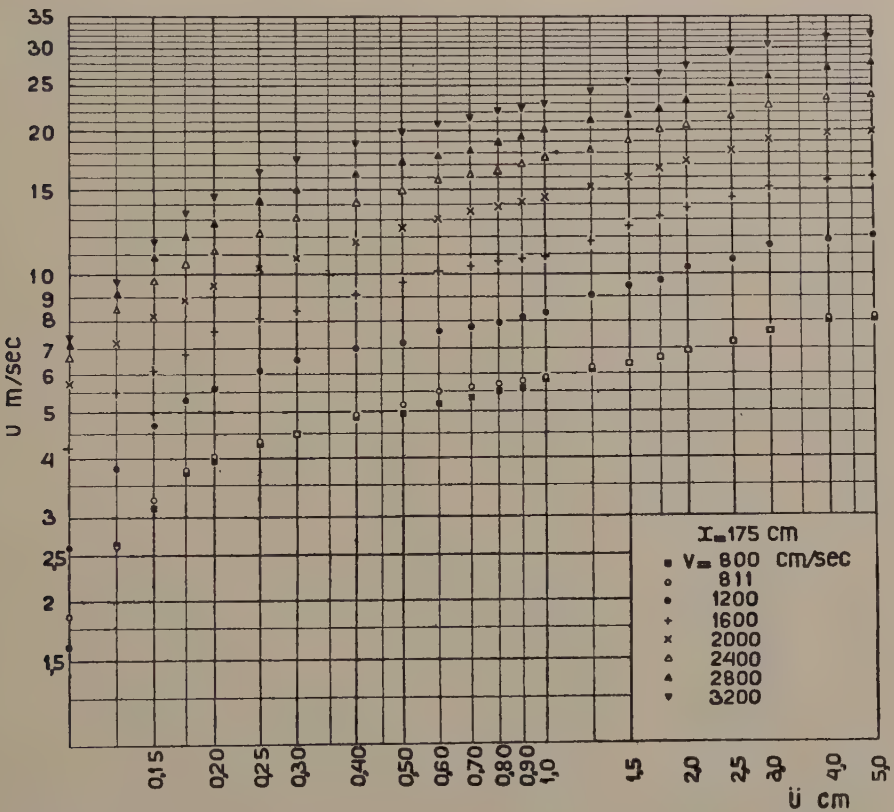

Fig. 2. Logarithmic scale diagram of the velocity in the boundary layer with $x = 175$ cm.

whether the velocity in the boundary layer could be represented by the formula $u = V\left(\frac{y}{\delta}\right)^n$ with a reasonable degree of accuracy. This proved to be the case; however, for all series the observed values of u seem to be arranged rather wavingly along the mean straight line[9]). This circumstance renders it difficult to determine the value of n with sufficient accuracy. The mean values, deduced from these diagrams, have been collected in table II.

They show a clearly marked decrease when x increases; probably this is due to the fact that the turbulent state of motion is not yet fully developed when x is small. On the contrary, n seems to be independent

9) The same phenomenon, although less pronounced. has been observed in the researches on a smooth surface, mentioned in note 6) (p. 28, fig. 8).

TABLE II. Values of n.

x	V cm/sec									Mean
	800 (v)	811 (v)	811 (t)	1200 (v)	1600 (v)	2000 (v)	2400 (v)	2800 (v)	3200 (v)	
15	—	0.33	0.38	—	—	—	—	—	—	0.35
20	—	0.34	0.37	—	—	—	—	—	—	0.35
25	—	0.32	0.36	0.34	0.37	0.36	0.36	0.38	0.37	0.36
37.5	—	0.30	0.32	—	—	—	—	—	—	0.31
50	—	0.30	0.31	0.33	0.34	0.33	0.34	0.33	0.33	0.33
62.5	—	0.285	0.31	—	—	—	—	—	—	0.30
75	0.27	0.275	0.29	0.27	0.27	0.27	0.27	0.27	0.27	0.27
87.5	—	0.275	0.28	—	—	—	—	—	—	0.28
100	—	0.27	0.28	0.29	0.30	0.29	0.28	0.29	0.29	0.29
125	—	0.24	0.25	0.24	0.27	0.24	0.24	0.24	0.26	0.25
150	—	0.25	0.24	0.25	0.26	0.24	0.25	0.245	0.25	0.25
175	0.24	0.24	0.24	0.25	0.24	0.25	0.25	0.25	0.25	0.25

TABLE III. Values of δ (in cm).

x cm	V cm/sec								Mean
	800	811	1200	1600	2000	2400	2800	3200	
5	—	0.4	—	—	—	—	—	—	0.4
10	—	0.63	—	—	—	—	—	—	0.63
15	—	0.90	—	—	—	—	—	—	0.90
20	—	1.00	—	—	—	—	—	—	1.00
25	—	1.20	1.16	1.14	1.10	1.06	1.02	1.08	1.11
37.5	—	1.50	—	—	—	—	—	—	1.50
50	—	2.00	1.90	1.80	1.96	1.80	1.64	1.76	1.84
62.5	—	2.18	—	—	—	—	—	—	2.18
75	2.55	2.55	2.55	3.1	2.7	2.5	2.55	2.45	2.62
87.5	—	3.00	—	—	—	—	—	—	3.00
100	—	3.20	3.20	3.0	2.85	2.9	2.9	3.0	3.01
125	—	3.75	3.80	4.0	4.0	3.5	3.7	3.1	3.69
150	—	3.70	4.0	4.0	4.1	4.0	3.5	3.3	3.80
175	3.80	3.80	4.0	3.8	3.6	3.7	3.65	3.5	3.74

of V. With $x \geqslant 100$ cm, n becomes about 0.25. Our researches are not sufficient to prove whether this is the limiting value, or whether n will decrease still further when x increases to the value $^1/_7$ which holds for a smooth surface [10]).

The logarithmic diagrams give at the same time the thickness δ of the boundary layer; the results have been collected in table III (all numbers mentioned here relate to the series v).

The value of δ decreases, though not much, when V increases, contrary to what has been supposed in § 2. However, δ never can be determined with great accuracy and the sensibility of the hot wire decreases at higher values of V.

From a numerical integration of the value of $u\,(V-u)$ over the thickness of the boundary layer the defect of momentum I has been determined, according to form. (9). Writing:

$$c_i = \frac{I}{^1/_2\,\varrho V^2 x} = \frac{2I}{\varrho V^2 x} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

the coefficient c_i has the values given in table IV (again for the series v):

TABLE IV. Values of $c_i \times 1000 = \dfrac{2I}{\varrho V^2 x} \times 1000.$

x	V cm/sec								Mean
	800	811	1200	1600	2000	2400	2800	3200	
5	—	21.6	—	—	—	—	—	—	21.6
10	—	15.8	—	—	—	—	—	—	15.8
15	—	15.8	—	—	—	—	—	—	15.8
20	—	14.0	—	—	—	—	—	—	14.0
25	—	12.7	13.2	13.76	13.4	12.56	12.60	13.62	13.12
37.5	—	10.1	—	—	—	—	—	—	10.1
50	—	10.6	11.1	11.48	12.18	11.30	10.48	10.80	11.13
62.5	—	8.4	—	—	—	—	—	—	8.4
75	9.08	9.46	9.30	10.12	9.50	8.64	8.86	8.66	9.22
87.5	—	8.90	—	—	—	—	—	—	8.90
100	—	8.60	8.60	8.48	7.84	7.80	8.40	8.32	8.29
125	—	7.39	7.10	7.40	7.40	6.80	7.94	7.00	7.29
150	—	6.52	6.72	6.84	6.98	6.68	6.52	6.08	6.62
175	5.68	5.68	5.84	5.92	5.48	5.56	6.04	5.66	5.73

These values do not vary systematically with V.

10) Compare the papers mentioned in notes 4), (p. 238) and 6) (fig. 7 and 8).

§ 5. *Discussion of the results.*

It was of importance to compare the experimental results collected in tables II to IV with the formulae deduced in § 2.

In the first place this has been done with the exponent n. Writing form. (11) by means of (9) :

$$c_i = \frac{2n}{(2n+1)(n+1)} \frac{\delta}{x} \quad \ldots \ldots \ldots \quad (12)$$

and taking the mean values of δ and c_i for every value of x, we arrive at the following results for the expression $\frac{n}{(2n+1)(n+1)}$:

TABLE V.

x	δ	δ/x	c_i	$\frac{(2n+1)(n+1)}{n}$
25	1.11	0.0444	0.01312	7.77
50	1.84	0.0368	0.01140	6.63
75	2.63	0.0351	0.00922	7.61
100	3.01	0.0301	0.00830	7.25
125	3.69	0.0295	0.00748	7.89
150	3.80	0.0253	0.00662	7.64
175	3.72	0.0213	0.00574	7.42
				Mean... 7.44

This gives for n about 0.25 (exactly 0.255), which is in agreement with the value deduced from the measurements on the velocity distribution for the higher values of x.

If this value of n is valid over the whole surface, form. (6) will become :

$$\delta = 5{,}02\, c^{2/5} \left(\frac{x}{h}\right)^{2/5} (1 - 1{,}75\, \beta x)$$

Now it was found in the experiments on waffle plate II that the velocity outside of the boundary layer increased by 160 cm/sec over the full length (198 cm) of the model, when V_0 was equal to 2400 cm/sec. This leads to $\beta = 0.000337$. If we accept that the same value of β holds for all series the factor $(1—1.75\, \beta\, x)$ will become equal to $1—0.00059\, x$.

Now $h = 0.17$ cm ; therefore we get for the constant the values mentioned in table VI :

TABLE VI.

x	δ	$c^{2/3}$	δ calculated from form. (13)
25	1.11	0.0473	1.21
50	1.84	0.0493	1.89
75	2.63	0.0555	2.44
100	3.01	0.0531	2.92
125	3.69	0.0571	3.32
150	3.80	0.0527	3.70
175	3.72	0.0459	4.03
	Mean...	0.0516	

With the mean value 0.0516, the expression for δ becomes

$$\delta = 0{,}259 \left(\frac{x}{h}\right)^{2/3} h\,(1 - 0{,}00059\,x) \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The values of δ derived from this formula have also been mentioned in table VI. From form. (12) we now find for c_i :

$$c_i = 0{,}0691 \left(\frac{x}{h}\right)^{-1/3} (1 - 0{,}00059\,x) \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Table VII gives the comparison between the values of c_i calculated from this expression and those derived from the observed values of I.

TABLE VII.

x	(c_i) calculated	(c_i) experimental
25	0.0130	0.01312
50	0.0111	0.01140
75	0.00871	0.00922
100	0.00779	0.00829
125	0.00709	0.00729
150	0.00660	0.00662
175	0.00617	0.00573

Although the differences between both sets of numbers are important, it may be said that in general the results found for n (as deduced from the

logarithmic diagrams for the velocity distribution), for the thickness of the boundary layer and for the defect of momentum in the boundary layer. confirm the relations deduced in § 2.

With the value of the constant mentioned before, form. (3) gives for τ_0:

$$\tau_0 = 0{,}01'7\,\varrho V^2 \sqrt{\frac{h}{\delta}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

The resistance experienced by a single wall of length l and breadth b, in a flow with the constant velocity V, now becomes according to (8) (with $\beta = 0$):

$$W = 0{,}0690 \left(\frac{h}{x}\right)^{1/8} \tfrac{1}{2} \varrho V^2\, bx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

§ 6. *Measurements on the velocity distribution along plate II.*

As stated in § 3, two series of measurements on the velocity distribution in the boundary layer along plate II were performed in order to check the results found with plate I.

The plate was mounted in the same way as mentioned at the end of § 3, with the leading edge at $X = 129$ cm behind the honey comb of the wind tunnel. The measurements were carried out at a distance of $x = 198$ cm from the leading edge, starting from a valley; a second series was performed 3 mm down-stream starting from a top. The hot wire anemometer was put at 40 cm above the bottom of the tunnel. Measurements have been performed only for the value $V_p = 2400$ cm/sec (Pitot-tube 42 cm behind honey comb; 52 cm from the front wall and 20 cm above the tunnel bottom), in which case V_x proved to be 2560 cm/sec according to the readings of the hot wire, which instrument had been calibrated carefully before. This leads to the velocity increase of 160 cm/sec along the surface, as stated before.

The results have been collected in the last columns of table I. They prove, as was the case with plate I at the higher values of x, that the velocity in the boundary layer increases with the 0.25-power of y; here too in a logarithmic diagram the observed values of the velocity are grouped more or less wavingly along the mean straight line. The exact evaluation of n and δ becomes rather difficult on account of this phenomenon.

The value of δ proved to be about 4.8 cm.

§ 7. *Determination of the resistance of plate II.*

The total resistance of the model, as found by means of the balance, has been represented in fig. 3 as a function of V_0 ($+ \ldots\ldots +$); in the same diagram the suction at the trailing edge ($\times \ldots\ldots \times$) and the resis-

tance of the suspension wires (d) have been given. When the total resistance is diminished by the suction and by the resistance of the

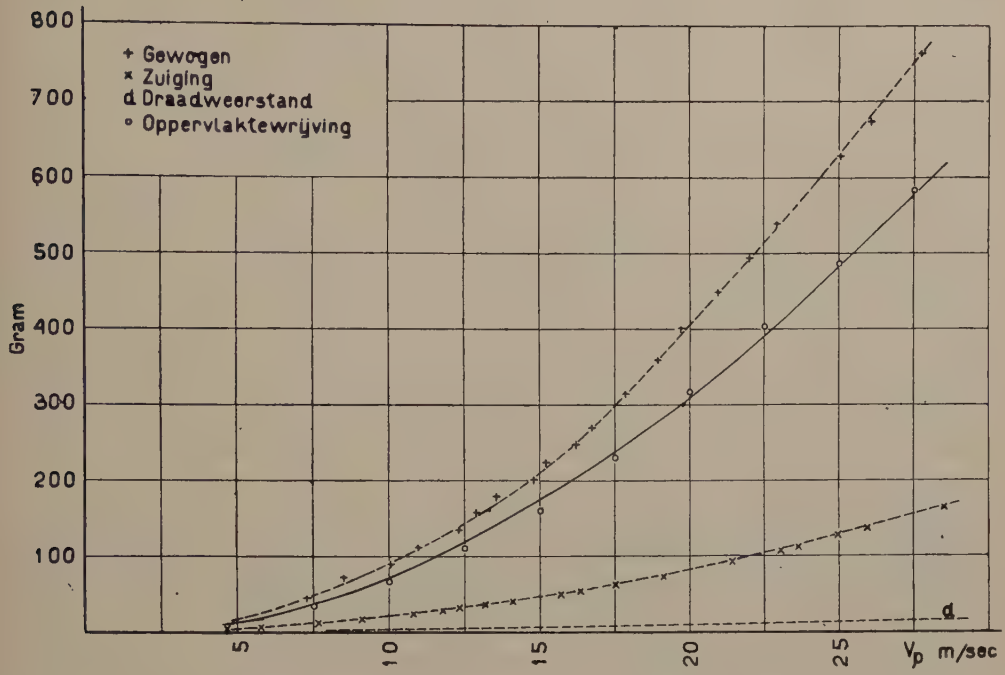

Fig. 3. Total resistance and surface friction as function of the velocity.

suspension wires we get the surface friction. From the diagram the following values have been deduced for this surface friction (∘ . . . . ∘).

TABLE VIII.

V_0 m/sec	Surface friction (both sides), in grammes	V_0 m/sec	Surface friction (both sides), in grammes
5	12	17.5	230
7.5	34	20	316
10	66	22.5	404
12.5	110	25	487
15	160	27.5	584

They may be represented by the interpolation formula:

$$W = 0{,}77\, V_0^2 \quad (V_0 \text{ in m/sec}),$$

which is in agreement with the supposition that the resistance is proportional to the square of the velocity. This formula is represented in fig. 3 by the full drawn line.

The results of the weighing have now been compared with the value of

the resistance as deduced from the experiments on the velocity distribution in the boundary layer. To this end the value of

$$I = \varrho \int_0^\delta u\,(V - u)\,dy$$

has been calculated from the measurements mentioned in § 6.

It has to be born in mind that the velocity in the first part of the boundary layer cannot be found with sufficient accuracy. Starting from a valley f.i. we might expect that the hot wire anemometer, which is at first more or less screened by the pyramids, will be influenced at higher values of y by the v- and w-components of the velocity. On the other hand, when we start from a "top", and call the distance y between the hot wire and the surface zero when the wire just touches the pyramids, as has been done before, we still have to take into account the flow between top and valley, which region of course was not accessible with the apparatus used by us. We have allowed ourselves to extrapolate the u-y-curve from top to valley i.e. over the distance h until the value $u = 0$ is reached at the base of a pyramid. The region between top and valley accounts for about 4 % of the total defect of momentum, and therefore an estimation of the flow in this part of the boundary layer will not lead to important errors.

In this way we found for the expression $\dfrac{I}{\varrho} = \displaystyle\int_0^\delta u\,(V - u)\,dy$ when starting from a valley: 3469000 cm^3/sec^2, and when starting from a top, taking the flow between top and valley into account, 3242000 cm^3/sec^2 (as mentioned in § 6, V_p was 2400 cm/sec). The difference between the two values is due to the region of high velocities (i.e. high values of y), where the smaller sensibility of the hot wire anemometer leads to less accurate readings of the velocity than in the region of low air speed near the surface.

The mean value of $\dfrac{I}{\varrho} = 3346000$ cm^3/sec^2 gives for the surface friction on both sides of the plate together at $V_p = 2400$ cm/sec, according to form. (10) with $\varrho = \dfrac{1}{8000}$ and $b = 50$ cm:

$$W = \frac{3346000}{8000}\,(1 + 0{,}9\,\beta x) = 444 \text{ grammes}$$

when we suppose that the velocity distribution is the same for all heights.

The surface friction as determined by weighing is, according to fig. 3, at $V_p = 2400$ cm/sec : 454 grammes, which differs little from the value deduced from the measurements on the velocity distribution in the boundary layer.

It is of importance to compare this value of W with that given by the

formula deduced from the measurements with plate I, taking x equal to 198 cm and $h=0.15$ cm. Form. (13) leads in this case to:

$$\delta = 4{,}13 \text{ cm}$$

hence form. (9) gives:

$$\frac{I}{\varrho}=\frac{4{,}13}{7{,}5}(V_{198})^2=\frac{4{,}13}{7{,}5}(1{,}066)^2\, V_0^2=0{,}626\, V_0^2,$$

and form. (10):

$$W=0{,}83\, V_0^2 \;(V_0 \text{ in m/sec}),$$

which leads to $W=478$ gram at $V_0=24$ m/sec. The check must be considered fair, especially as it is not easy to determine h with sufficient accuracy.

§ 8. *Resistance deduced from total head loss in the wake.*

Moreover the resistance experienced by the model has been evaluated from the total head loss of the flow in the wake of the model. We suppose the flow to be stationary and will neglect the components of the velocity perpendicular to the direction of the undisturbed wind as the model is a long, thin board, mounted in the direction of the flow. Calling the static pressure in a section of the tunnel in front of the model p_A and the static pressure in a section behind the model p_B we get as the resultant of the pressure forces for a section of unit height:

$$W_p=\int (p_A - p_B)\, dy.$$

The change of momentum per unit time of the air passing the sections A and B is found from the total head loss. The mass of air entering A per unit time is $\int \varrho u_A\, dy$; its momentum: $\int \varrho u_A^2\, dy$; the momentum of the air leaving B is: $\int \varrho u_B^2\, dy$. Consequently:

$$W=\int (p_A+\varrho u_A^2)\, dy-\int (p_B+\varrho u_B^2)\, dy \quad . \quad . \quad . \quad . \quad . \quad (17)$$

By introducing the total head: $H=p+\frac{1}{2}\varrho u^2$, (17) is reduced to:

$$W=\int \triangle H\, dy+\int {}^1/_2\, \varrho u_A^2\, dy-\int {}^1/_2\, \varrho u_B^2\, dy \quad . \quad . \quad . \quad . \quad (18)$$

The second and third members at the right hand side of (18) are nearly equal; taken together they can be treated as a correction applied to the first term. They can be simplified when u_A and u_B are written as the

sum of the undisturbed velocity V and a disturbing velocity u_a, u_b; in this way the correction becomes:

$$\left.\begin{aligned}\int {}^1/_2\,\varrho\,(V^2 + 2u_aV + u_a^2)\,dy - \int {}^1/_2\,\varrho\,(V^2 + 2u_bV + u_b^2)\,dy = \\ = \varrho V \int (u_a - u_b)\,dy + {}^1/_2\,\varrho \int (u_a^2 - u_b^2)\,dy\end{aligned}\right\} . \quad (19)$$

The equation of continuity, applied to the region between A and B, proves that the first term at the right hand side of (19) is zero. Putting $u_A = V =$ constant which holds if the section A is taken far enough in front of the model and the flow is measured not too near to the honey comb, u_a will be zero, and (18) is reduced to:

$$W = \int \triangle H\,dy - {}^1/_2\,\varrho \int u_b^2\,dy \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where the integral is extended only over the region of turbulent motion in the wake.

In evaluating the experimental data, the correction proved to be of the order of 10 % of the uncorrected value of W. On the other hand it was found that the values of H varied in a rather important and unsystematic manner at various heights above the tunnel bottom. On account of this it was considered superfluous to calculate the correction in a more refined manner.

The section B was chosen 40 cm behind the trailing edge of the model, while the values of u_A and H_A were derived from a Pitot-tube put at 42 cm behind the honey comb, 20 cm above the bottom of the tunnel and 52 cm from the front wall. The values in the section B were determined by means of a Pitot-tube which could be shifted perpendicularly to the plate. In order to observe the variations of $\triangle H$ with the height, H_B was read at the following values of z: 12, 15, 17, 20, 30, 40, 44, 48, 52, 56, 60, 63, 65 and 68 cm (measured from the tunnel bottom). In a direction perpendicular to the plate the Pitot-tube was set at the following values of y: 20, 22, 23, 24, 25, 26, 28, 30, 31, 32, 33, 34, 34½, 35, 35½, 36, 36½, 37, 37½, 38, 38½, 39, 39½, 40, 40½, 41, 41½, 42, 42½, 43, 43½, 44, 44½, 45, 45½, 46, 47, 48, 49, 50, 52, 54, 55, 56, 57, 58, 60 cm (measured from the front wall).

During these experiments the model was fixed by means of thin steel wires, in the same way as during the measurements with the balance. On account of the dimensions of the tunnel and of the position of the section B, the distance X had to be reduced to 129 cm; the distance from the model to the vertical front wall was 41.5 cm, measured from half way the thickness of the board.

While reading H_B the velocity of the air in the tunnel as indicated by Pitot-tube A, was kept constant at $V_p = 2400$ cm/sec. At every position

of B 10 readings were taken of the differences of the static pressures and of those of the dynamic pressures $(p + \frac{1}{2} \varrho V^2)$ on A and B, at intervals of 5 seconds. A third manometer connected to A allowed the determination of the velocity V_p. When the pressures in a certain position had been observed, the Pitot-tube B was shifted to the next position and again 10 readings were taken; etc. During the readings at a constant value of z the fan of the wind tunnel was kept in action; it had to be stopped when z was modified.

The values of $\triangle H$ were plotted as function of y for every value of z; the areas of the curves obtained in this way were determined by means of a planimeter. In first approximation the area J of each of them represents the resistance of the model per cm height at the particular value of z at which $\triangle H$ has been determined. Then the correction

$$C = \int \tfrac{1}{2} \varrho \, u_b^2 \, dy$$

was deduced from $\triangle H$ and $\triangle p$ for the same values of z. Taking this correction into account, so that $J_{corr.} = J - C$, the area of the curve for $J_{corr.}$ as a function of z gives the resistance of the whole model. The results are collected in table IX (J being expressed in mm water $\times$ cm, i.e. in kg/m^2 $\times$ cm).

TABLE IX. Total head loss in the wake of waffle plate II.

z cm	12	15	17	20	30	40	44	48	52	56	60	63	65	68
J mm.cm	18.3	104.7	116.2	111.6	138.9	145.0	132.5	129.8	127.9	107.4	91.4	98.8	97.6	12.3
Correction	?	6.6	8.8	7.9	12.3	14.0	11.2	10.7	10.6	7.8	5.7	6.3	4.8	?
J corr.	18.3	98.1	107.4	103.7	126.6	131.0	121.3	119.1	117.3	99.6	85.7	91.5	92.8	12.3

Integrating the value of $J_{corr.}$ with respect to z, we find:

$$W = \int J_{corr.} \, dz = 561{,}5 \text{ grammes.}$$

Taking into account the suction at the trailing edge, the value of which is 120 grammes for $V_p = 2400$ cm/sec, the surface friction on both sides of the model will be 441.5 grammes.

Notwithstanding the uncertainty in the determination of the resistance from the total head loss, the agreement between the resistance found in this way and that deduced from other methods seems to be fair.

§ 9. *Summary.*

From measurements of the distribution of the velocity in the boundary layer the value of the exponent n in the relation $u = V\left(\frac{y}{\delta}\right)^n$ has been

deduced. This exponent proves to be independent of the velocity V and seems to be determined entirely by geometrical relations. The results provisionally lead to the conclusion that n approaches to the limit of 0.25 for increasing values of δ.

The value of the thickness of the boundary layer δ deduced from the experiments has been compared with the relations found in § 2 for the fully developed turbulent state of motion along rough surfaces. As may be expected from the formulae, δ does not depend on V and is determined entirely by x, by the height h of the pyramids on the surface, and by a numerical factor. This factor, which also occurs in the formula for the resistance, is connected with the shape and the arrangement of the pyramids on the surface. It may be expected that the relations for δ, for the shearing stresses near the surface and for the resistance coefficient deduced from our measurements, will hold for a series of rough surfaces similar to those investigated. The experimental material, however, is not sufficient to decide whether this supposition is of general validity.

The surface friction of both sides of plate II, as weighed on the balance, is in agreement with the resistance as determined from the total head loss in the wake by means of a Pitot-tube; the surface friction deduced from the loss of momentum in the boundary layer (determined with a hot wire anemometer) leads to the same value. The measurements with the balance lead to the formula:

$$W = 0{,}77\, V_0^2 \; (V_0 \text{ in m/sec}),$$

while form. (10) deduced from the measurements performed at plate I, inserting $\beta = 0.000337$, gives:

$$W \cong 0{,}83\, V_0^2.$$

The results of the determinations of the surface friction of both sides together according to several methods may be summarized as follows (the length of the board being 198 cm, its breadth 50 cm and the velocity $V_0 = 2400$ cm/sec):

Plate II, from measurements with the balance	454 gr.
from total head loss in the wake	441 gr.
from loss of momentum in the boundary layer	444 gr.
from form. (8), $x = 198$, $h = 0.15$ cm, $\beta = 0.000337$	478 gr.

Mathematics. — *On the Motion of a Plane Fixed System with Two Degrees of Freedom.* (Second Communication [1])). By Prof. W. VAN DER WOUDE.

(Communicated at the meeting of March 31, 1928).

§ 1. By the motion of a fixed system we always understand, at least in purely kinematical considerations, the motion of two fixed systems relative to each other; already in the usual indication of the problem we notice the peculiar lack of symmetry that strikes us in the further treatment [2]). In the simplest case, where this motion depends on one parameter, this lack is very little troublesome; it seems to me that already in the next case — the motion depending on two parameters — it is certainly worth while to pass on to a more symmetrical representation.

As in this case we choose a system of axes that is not fixed to either of the systems, the formulas are in the beginning slightly more complicated than in the usual method; this disadvantage disappears, however, as soon as we give this system of axes the movement that is prescribed by the problem so that the symmetry remains intact.

The followed method is briefly this: With the exception of a few special cases (§ 3) there always exists a definite line d, that is not fixed to either of the systems and that is the locus of the possible poles of rotation; this line may be chosen as the X-axis of a system of axes. It is then obvious that a definite point must be chosen as origin. In this way the formulas for the motion have become so simple that the known conclusions may be read at once.

For the sake of a more outward consequence the usual expressions "fixed" and "movable system" have been replaced by "the systems Σ_1 and Σ_2".

§ 2. Let OXY be a rectangular system of axes; the coordinates (x, y) of any point always relate to *this* system; whenever there is question of the components of a vector we always mean the projections of this

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1) The earlier communication (these Proceedings, Vol. **29**, p. 652) gives a list of literature. I received for perusal an article on this subject of Dr. H. J. E. BETH, which will appear in the next number of the Nieuw Archief voor Wiskunde (2e reeks, deel XV, vierde stuk). The method of Dr. BETH, however, is entirely different from mine. Dr. BETH has always treated also the non holonomous cases; I restrict myself in this paper to the holonomous cases, although an extension would not be difficult.

2) Except in a few chapters of:
R. BRICARD. Leçons de Cinématique. Tome I. Paris, Gauthier—Villars; 1926.

vector on *this* system. Let further Σ_1 and Σ_2 be two plane fixed systems the motion of which relative to each other depends on two parameters u and v; we shall assume that the motion of both relative to OXY also depends on u and v, that, however, it is possible that the motion of one, e.g. of Σ_1, depends on one parameter, e.g. of u, but that in this case the motion of Σ_2 relative to OXY depends either on both u and v or on v only.

For convenience' sake there follows here first a summary of the formulas that express the displacements occurring in this motion. The elementary displacement relative to OXY of a point (x, y) that is fixed to Σ_i $(i = 1, 2)$ is defined by

$$\left.\begin{aligned} \delta x &= (\xi_1^{(i)} - \omega_1^{(i)} y)\,du + (\xi_2^{(i)} - \omega_2^{(i)} y)\,dv \\ \delta y &= (\eta_1^{(i)} + \omega_1^{(i)} x)\,du + (\eta_2^{(i)} + \omega_2^{(i)} x)\,dv \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Here

$$\xi_1^{(i)},\ \xi_2^{(i)},\ \eta_1^{(i)},\ \eta_2^{(i)},\ \omega_1^{(i)},\ \omega_2^{(i)}$$

have the well known signification; between these quantities exist the relations [1])

$$\frac{\partial \omega_1^{(i)}}{\partial v} = \frac{\partial \omega_2^{(i)}}{\partial u} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2^{\alpha})$$

$$\left.\begin{aligned} \frac{\partial \xi_1^{(i)}}{\partial v} - \frac{\partial \xi_2^{(i)}}{\partial u} &= \eta_2^{(i)}\,\omega_1^{(i)} - \eta_1^{(i)}\,\omega_2^{(i)} \\ \frac{\partial \eta_1^{(i)}}{\partial v} - \frac{\partial \eta_2^{(i)}}{\partial u} &= -\,\xi_2^{(i)}\,\omega_1^{(i)} + \xi_1^{(i)}\omega_2^{(i)} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2^{\beta})$$

In the same way the elementary displacement relative to Σ_1 of a point (x, y) that is fixed to Σ_2 is given by

$$\left.\begin{aligned} \delta x &= (\xi_1 - \omega_1 y)\,du + (\xi_2 - \omega_2 y)\,dv \\ \delta y &= (\eta_1 + \omega_1 x)\,du + (\eta_2 + \omega_2 x)\,dv \end{aligned}\right\}; \quad . \quad . \quad . \quad . \quad (3)$$

ω_1 and ω_2 are the rotations of Σ_2 in its motion relative to Σ_1; ξ_1 is the projection on OX of the vector that expresses the velocity relative to Σ_1 of the point (o, o) fixed to Σ_2. If we consider (x, y) as a point that is fixed to Σ_1 and if in (3) we replace ξ_1 by $-\xi_1$ etc. the displacement relative to Σ_2 of a point fixed to Σ_1 is expressed by these equations.

In this case it follows from (1) and (3) that

$$\left.\begin{aligned} \xi_1 = \xi_1^{(2)} - \xi_1^{(1)};\ \xi_2 = \xi_2^{(2)} - \xi_2^{(1)};\ \eta_1 = \eta_1^{(2)} - \eta_1^{1};\ \eta_2 = \eta_2^{2} - \eta_2^{(1)} \\ \omega_1 = \omega_1^{(2)} - \omega_1^{(1)};\ \omega_2 = \omega_2^{(2)} - \omega_2^{(1)} \end{aligned}\right\} \quad . \quad (4)$$

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[1]) Cf. e.g. G. DARBOUX: Théorie des Surfaces I, p. 67, 71 (Gauthier—Villars, Paris), or L. P. EISENHART: A Treatise on Differential Geometry, p. 168, 170 (Ginn and Co., Boston, New York, London). Our formulas (2) however are only identical with the cited ones when we replace $\xi_1^{(i)}$ by $-\xi_1^{(i)}$ etc. as l.c. always the inverse motion, the motion of OXY relative to Σ_1, is considered.

The equations (2^{β}) do not hold good for $\xi_1 \ldots \omega_2$. (2^{α}) remains valid; we can see this by filling in $i=1$ and $i=2$ in (2^{α}) and by subtracting the two equations from each other. It appears that

$$\frac{\partial \omega_1}{\partial v} = \frac{\partial \omega_2}{\partial u} \quad \ldots \ldots \ldots \ldots (5)$$

§ 3. We start from (3) and we consider, therefore, the motion of Σ_2 relative to Σ_1. We shall call the possible movements depending on two parameters the system $[\mathfrak{M}^2]$.

The locus of the poles is found from:

$$\delta x = \delta y = 0$$

and has, therefore, as equation

$$\left\| \begin{matrix} \xi_1 - \omega_1 y & \xi_2 - \omega_2 y \\ \eta_1 + \omega_1 x & \eta_2 + \omega_2 x \end{matrix} \right\| = 0$$

or

$$(\xi_1 \omega_2 - \xi_2 \omega_1) x + (\eta_1 \omega_2 - \eta_2 \omega_1) y + \xi_1 \eta_2 - \xi_2 \eta_1 = 0$$

If for the moment we exclude the cases where

$$\omega_1 = \omega_2 = 0$$

or

$$\left\| \begin{matrix} \xi_1 \, \eta_1 \, \omega_1 \\ \xi_2 \, \eta_2 \, \omega_2 \end{matrix} \right\| = 0,$$

(6) represents a straight line d.

It is evident that

$$\omega_1 = \omega_2 = 0$$

means that $[\mathfrak{M}^2]$ contains only translations and that

$$\left\| \begin{matrix} \xi_1 \, \eta_1 \, \omega_1 \\ \xi_2 \, \eta_2 \, \omega_2 \end{matrix} \right\| = 0$$

indicates that the system $[\mathfrak{M}^2]$ depends on one parameter only.

In the future we shall always exclude these cases.

§ 4. We shall now make the condition that d coincide with OX; for this it is necessary and sufficient that

$$\xi_1 = \xi_2 = 0$$

For the sake of a further simplification we first remark that owing to (5) we can introduce a new variable θ through

$$2\, d\theta = \omega_1\, du + \omega_2\, dv \quad \ldots \ldots \ldots (7^a)$$

We denote an integrating factor of $\eta_1\, du + \eta_2\, dv$ by $\dfrac{1}{2H(u, v)}$ so that we can put

$$2H\, d\tau = \eta_1\, du + \eta_2\, dv \quad \ldots \ldots \ldots (7^b)$$

We shall further introduce in H the variables θ and τ defined through (7^a) and (7^b) but for a constant, in stead of u and v.

The displacement relative to Σ_1 of a point (x, y) of Σ_2 is now expressed by

$$\left.\begin{aligned} \delta x &= \qquad\quad -2\,y\,d\theta \\ \delta y &= 2\,H\,d\tau + 2\,x\,d\theta \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (I)$$

It is evident that $d\theta = 0$ means a translation and $d\tau = 0$ a rotation round the origin; also that θ is twice the angle between two lines one of which is fixed to Σ_1 and the other to Σ_2 and that τ may be replaced by any function of τ without the form of (I) changing.

We shall now represent the displacement relative to OXY of a point $(x\,y)$ that is fixed to Σ_1 by

$$\left.\begin{aligned} \delta x &= (U_1 - \Omega_1\,y)\,d\tau + [U_2 - (\Omega_2 - 1)\,y]\,d\theta \\ \delta y &= (V_1 - H + \Omega_1 x)\,d\tau + [V_2 + (\Omega_2 - 1)\,x]\,d\theta \end{aligned}\right\}; \quad \ldots \quad (8)$$

accordingly for the displacement relative to OXY of a point of Σ_2 we have

$$\left.\begin{aligned} \delta x &= (U_1 - \Omega_1\,y)\,d\tau + [U_2 - (\Omega_2 + 1)\,y]\,d\theta \\ \delta y &= (V_1 + H + \Omega_1 x)\,d\tau + [V_2 + [\Omega_2 + 1)\,x]\,d\theta \end{aligned}\right\}. \quad \ldots \quad (9)$$

From (8) as well as from (9) there follow relations (see 2^a and 2^3) between U_1, U_2, V_1, V_2, Ω_1, Ω_2 and H; through addition and subtraction these are simplified to

$$\left.\begin{aligned} &\frac{\partial \Omega_1}{\partial \theta} = \frac{\partial \Omega_2}{\partial \tau} && \frac{\partial U_1}{\partial \theta} - \frac{\partial U_2}{\partial \tau} = V_2\,\Omega_1 - V_1\,\Omega_2 - H \\ &V_1 + H\Omega_2 = 0 && \\ &\frac{\partial H}{\partial \theta} = U_1 && \frac{\partial V_1}{\partial \theta} - \frac{\partial V_2}{\partial \tau} = U_1\,\Omega_2 - U_2\,\Omega_1 \end{aligned}\right\} \quad \ldots \quad (10)$$

§ 5. Let a definite displacement out of $[\mathfrak{M}^2]$ be defined by

$$\frac{d\theta}{d\tau} = \lambda\,;$$

if $(x, 0)$ is the pole of rotation for the motion of Σ_1 and Σ_2 relative to each other, we have

$$H + \lambda\,x = 0$$

Now in the system Σ_1 d turns about the point $(x', 0)$ for which in (9)

$$\delta\,y = 0$$

hence about the point that is defined by

$$V_1 - H + \Omega_1\,x' + \lambda\,[V_2 + (\Omega_2 - 1)\,x'] = 0\,;$$

in the system Σ_2 d turns about the point $(x'', 0)$ that is defined by

$$V_1 + H + \Omega_1\,x'' + \lambda\,[V_2 + (\Omega_2 + 1)\,x''] = 0.$$

The former two points coincide when

$$\Omega_1 x^2 + (V_1 - \Omega_2 H)x - V_2 H = 0 \quad . \quad . \quad . \quad . \quad . \quad (11)$$

if this is the case $(x, 0)$ and $(x'', 0)$ also coincide as might be expected. Through (11) two points are defined — at least if $\Omega_1 \pm 0$ —; we can call them the *stationary poles of rotation* (for the given position). For the moment we shall further put

$$\Omega_1 \neq 0. \, .$$

In the future we shall always choose the middle between these stationary poles of rotation as origin of our system of coordinates of which so far we had defined the X-axis, not the origin. Now we have always

$$V_1 - H\Omega_2 = 0.$$

In connection with one of the formulas (10) it follows from this that

$$V_1 = \Omega_2 = 0.$$

It is impossible that H is identically equal to zero as in this case the motion of Σ_1 and Σ_2 relative to each other would only have one degree of freedom.

The formulas (8), (9), and (10) are now greatly simplified. We have already found

$$V_1 = \Omega_2 = 0,$$

further in (10)

$$\frac{\partial \Omega_1}{\partial \theta} = 0$$

hence Ω_1 is a function of τ only. Accordingly we can again denote $\int \Omega_1 \, d\tau$ by a new variable; if this is again called τ we have — cf. (10) —

$$\frac{\partial H}{\partial \theta} = U_1 \, ; \; \frac{\partial V_2}{\partial \tau} = U_2$$

$$\frac{\partial U_1}{\partial \theta} - \frac{\partial U_2}{\partial \tau} = V_2 - H.$$

SUMMARISING. The displacement relative to Σ_1 of a point of Σ_2, hence any displacement out of $[\mathfrak{M}^2]$, is expressed by

$$\left. \begin{aligned} \delta x &= \qquad\quad -2\,y d\theta \\ \delta y &= 2\,H d\tau + 2\,x d\theta \end{aligned} \right\} ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (I)$$

the displacement relative to OXY of a point of Σ_1 by

$$\left. \begin{aligned} \delta x &= \left(\frac{\partial H}{\partial \theta} - y \right) d\tau + \left(\frac{\partial V_2}{\partial \tau} + y \right) d\theta \\ \delta y &= (-H + x)\, d\tau + (V_2 - x)\, d\theta \end{aligned} \right\} ; \quad . \quad . \quad . \quad . \quad (II)$$

the displacement relative to OXY of a point of Σ_2 by

$$\left.\begin{aligned} \delta x &= \left(\frac{\partial H}{\partial \theta} - y\right) d\tau + \left(\frac{\partial V_2}{\partial \tau} - y\right) d\theta \\ \delta y &= (H + x)\, d\tau + (V_2 + x)\, d\theta \end{aligned}\right\} \quad \ldots \ldots \quad (III)$$

Between the functions $H(\tau, \theta)$ and $V_2(\tau, \theta)$ there only exists the relation

$$\frac{\partial^2 H}{\partial \theta^2} - \frac{\partial^2 V_2}{\partial \tau^2} + H - V_2 = 0 \quad \ldots \ldots \ldots \quad (IV)$$

§ 6. *Simple Results.* Let any displacement be given by

$$\frac{d\theta}{d\tau} = \lambda.$$

The pole of rotation for the displacement of Σ_1 and Σ_2 relative to each other is the point $P\left(-\frac{H}{\lambda}, 0\right)$; in the plane Σ_1 d turns round the point $Q_1\left(\frac{H-\lambda V_2}{1-\lambda}, 0\right)$ for which

$$\delta y = 0;$$

in Σ_2 d turns round $Q_2\left(\frac{-H-\lambda V_2}{1+\lambda}, 0\right)$.

In any position there exists a projective correspondence between P and Q_1 and also between P and Q_2 [1]*.*

Special cases:

1. The three points coincide in the stationary poles of rotation.
2. If O is the pole of rotation V_2 and $-V_2$ are the abscissae of Q_1 and Q_2.
3. If Σ_1 and Σ_2 have a translation relative to each other (P lies at infinity; $\lambda = 0$), d turns round Q_1 (H, o) in Σ_1, round $(-H, o)$ in Σ_2.
4. If a has a translation in Σ_1, $(-H, o)$ is the pole of rotation; if d has a translation in Σ_2, (H, o) is the pole of rotation.

O is always the middle between the found pairs of points.

The motion of d in Σ_1 depends on one parameter only in the case that

$$-H + V_2 = 0$$

for then in (II)

$$\delta y = 0$$

for any point ($x = 0$) if the displacement is defined by

$$d\tau - d\theta = 0;$$

[1]) Cf. BETH l.c. who derives a complete classification of the movements with two parameters, including the non-holonomous ones, from the projective relations between P, Q_1 and Q_2.

in this case all points of d move on d and d is a fixed line in Σ_1. For any other displacement d turns in Σ_1 about the point (H, o).

But on the same condition the motion of d in Σ_2 depends on only one parameter; the displacement defined by

$$d\tau + d\theta = 0$$

leaves d at rest in Σ_2; for any other displacement it turns about the point $(-H, o)$. This gives the

THEOREM OF KOENIGS. *If the displacement of d in Σ_1 depends on only one parameter, this is also the case with the motion of d in Σ_2. The displacements that leave d at rest in Σ_1 and those that leave it at rest in Σ_2, are different.*

A second interesting special case is the following one. Suppose

$$V_2 = 0.$$

In this case the two stationary poles of rotation coincide in O; they correspond to the displacement for which

$$d\tau = 0.$$

In order to examine the displacement of O relative to Σ_1 and Σ_2 we have only to calculate δx and δy in (II) and (III) for $O\,(o, o)$ and to replace them by their opposites. Then evidently

$$\delta x = \delta y = 0.$$

The origin is accordingly at rest in Σ_1 and Σ_2, in other words:

If V_2 is identically equal to zero the system of movements $[\mathfrak{M}^2]$ of the systems Σ_1 and Σ_2 relative to each other contains a finite rotation about a point that is fixed to Σ_1 and to Σ_2.

This leads to the problem: when does the system $[\mathfrak{M}^2]$ contain finite rotations about a point that is fixed to Σ_1 and to Σ_2?

The pole of rotation must be fixed in Σ_1 and Σ_2: d always passes through the pole, hence d turns about the same point in Σ_1 and Σ_2, which point is accordingly one of the stationary poles of rotation. The problem is therefore: is it possible that for the *finite* movement defined by

$$\sqrt{H}\, d\tau + \sqrt{V_2}\, d\theta = 0$$

the pole of rotation $(\pm \sqrt{H V_2}, o)$ is fixed to Σ_1. The vector of the displacement of a point (x, y) relative to Σ_1 is given by the components $dx - \delta x,\ dy - \delta y$ when δx and δy are taken from (II) and if for $\dfrac{d\theta}{d\tau}$ $(\pm \sqrt{HV_2}, o)$ is substituted.

It is therefore necessary that for one of the points $(\pm \sqrt{H V_2}, o)$

$$dx - \delta x = 0$$

or

$$3\left(H\sqrt{V_2}\,\frac{\partial V_2}{\partial \tau} + V_2\sqrt{H}\,\frac{\partial H}{\partial \theta}\right) + V_2\sqrt{\theta_2}\,\frac{\partial H}{\partial \tau} + H\sqrt{H}\,\frac{\partial V_2}{\partial \theta} = 0.$$

If H and V_2 satisfy this equation besides (IV), $[\mathfrak{M}^2]$ *contains a finite rotation about a fixed point.*

As might be expected this equation is always satisfied if $V_2 = 0$.

Does the system $[\mathfrak{M}^2]$ contain *finite* rectilinear translations? The translation is always parallel to d (cf. (I)); however we have seen that d turns about a point at finite distance in Σ_1 as well as in Σ_2 and, accordingly, has neither a fixed direction in Σ_1 nor in Σ_2. A finite rectilinear translation can, therefore, only be expected in the case that we have excluded until now; in that case it is in fact contained in $[\mathfrak{M}^2]$ (cf. §§ 5, 7; $\Omega_1 = 0$).

§ 7. We shall now briefly discuss the case that has so far been excluded where (cf. § 5)

$$\Omega_1 = 0$$

If we suppose in the first place

$$V_1 - \Omega_2 H \neq 0 \,. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\alpha)$$

it appears from (II) § 5 that:

there is only one stationary pole of rotation (or the other one lies at infinity).

If we choose this stationary pole of rotation (not at infinity) as origin we have:

$$V_2 = 0.$$

It appears further from (10) that Ω_2 is a function of θ only and $\Omega_2 H^2$ of τ only, hence

$$H = \frac{\varphi(\tau)}{\sqrt{\Omega_2}}$$

We shall introduce $\int \varphi(\tau)\, d\tau$ as new variable; if we call this variable again τ the form of (I) does not change and H is a function of θ only. Further

$$\Omega_2 = \frac{1}{H^2}\,,\quad V_1 = -\frac{1}{H}\,,\quad U_1 = \frac{\partial H}{\partial \theta}.$$

Any displacement relative to Σ_1 of a point of Σ_2 is still expressed by

$$\left.\begin{aligned} \delta x &= \qquad\qquad\quad -2\,y\,d\theta \\ \delta y &= 2\,H(\theta)\,d\tau + 2\,x\,d\theta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{I})$$

the displacement relative to OXY of a point of Σ_1 by

$$\left.\begin{aligned} \delta x &= \quad \frac{\partial H}{\partial \theta}\,d\tau + \left[U_2 - \left(\frac{1}{H^2} - 1\right) y\right] d\theta \\ \delta y &= -\left(\frac{1}{H} + H\right) d\tau + \left(\frac{1}{H^2} - 1\right) x\,d\theta \end{aligned}\right\}; \quad . \quad . \quad . \quad . \quad (\mathrm{II})$$

the displacement relative to OXY of a point of Σ_2 by

$$\left.\begin{aligned} \delta x &= \frac{\partial H}{\partial \theta}\,d\tau + \left[U_2 - \left(\frac{1}{H^2}+1\right)y\right]d\theta \\ \delta y &= -\left(\frac{1}{H} - H\right)d\tau + \left(\frac{1}{H^2}+1\right)x\,d\theta \end{aligned}\right\} \quad \ldots \quad \text{(III)}$$

Between the functions U_2 and H there exists the relation

$$\frac{\partial^2 H}{\partial \theta^2} - \frac{\partial^2 U^2}{\partial \tau^2} + \frac{1}{H^3} + H = 0.$$

This proves: *if always (i. e. for any pair of values of u and v) $\Omega_1 = 0$, the system $[\mathfrak{M}^2]$ contains a rectilinear finite translation of Σ_1 and Σ_2 relative to each other, corresponding to $d\theta = 0$. If besides $H = \pm 1$ the motion of d in Σ_1 as well as in Σ_2 depends on one parameter only.*

There remain the possibilities

$$\Omega_1 = 0, \quad V_1 - \Omega_2 H = 0, \quad V_2 \neq 0 \quad \ldots \ldots \quad (\beta)$$

d does not contain any stationary pole of rotation.

$$\Omega_1 = 0, \quad V_1 - \Omega_2 H = 0, \quad V_2 = 0. \quad \ldots \ldots \quad (\gamma)$$

Any point of d is a stationary pole of rotation.

If, accordingly, Ω_1, $V_1 - \Omega_2 H$ and V_2 are always equal to zero, about any point of d a finite rotation is possible. I intend to come back to this remarkable case in a later short paper.

In the following discussion of the quantities of the second order this case $\Omega_1 = o$ is again excluded.

§ 8. *The quantities of the second order.*

If we start from a given original position, for a given $\frac{d\theta}{d\tau}$ the tangent to the path in Σ_1 of any point of Σ_2 is defined (and inversely); the quantities of the second order, e.g. the radius of curvature of any point in its path, are not defined before also $\frac{d^2\theta}{d\tau^2}$ is given.

We shall now put

$$\frac{d\theta}{d\tau} = \lambda, \quad \frac{d^2\theta}{d\tau^2} = \lambda'$$

and we shall only consider the system of infinitesimal displacements where λ is kept constant and λ' is variable. In other words, *we choose movements from* $[\mathfrak{M}^2]$ *with a fixed pole of rotation* where to any point a definite tangent to its path is already assigned.

For the present we assume that the movement of Σ_2 relative to Σ_1 is given by the functions ξ, η, ω that depend on one parameter t (the

time) and that the movement of Σ_1 relative to OXY is given by $\xi^{(1)}$, $\eta^{(1)}$, $\omega^{(1)}$ that also depend on t only. For the components of the velocity- and acceleration relative to Σ_1 of a point (x, y) of Σ_2 we have

$$v_x = \xi - \omega y, \quad v_y = \eta + \omega x.$$

$$J_x = \frac{dv_x}{dt} + \omega^{(1)} v_y = \frac{d\xi}{dt} + \omega^{(1)} \eta - (\eta^{(1)} + \eta)\, \omega - \frac{d\omega}{dt} y - \omega^2 x,$$

$$J_y = \frac{dv_x}{dt} - \omega^{(1)} v_x = \frac{d\eta}{dt} - \omega^{(1)} \xi + (\xi^{(1)} + \xi)\, \omega + \frac{d\omega}{dt} x - \omega^2 y$$

The radius of curvature of a point (x, y) of Σ_2 in its path in Σ_1 is given by

$$\frac{1}{R^2} = \frac{(V_x J_y - J_x V_y)^2}{V^6}$$

PROOF. If we use fixed axes we have

$$\frac{1}{R^2} = \frac{(x' y'' - y' x'')^2}{(x'^2 + y'^2)^3}$$

i.e. $\frac{1}{R^2}$ is equal to the square of the vector product of the velocity and the acceleration divided by the sixth power of the velocity. That is exactly what the above mentioned formula expresses.

In the same way it appears that the center of curvature in the path of $M(x, y)$ is the point

$$\mu \left(x - \frac{V_y}{V_x J_y - V_y J_x} V^2, \quad y + \frac{V_x}{V_x J_y - V_y J_x} V^2 \right)$$

We now pass on to the case in question by the following substitutions (cf. the formulas (I) and (II), § 5)

$$\xi = 0, \quad \eta = 2H, \quad \omega = 2\lambda$$

$$\xi^{(1)} = \frac{\partial H}{\partial \theta} + \lambda \frac{\partial V_2}{\partial \tau}, \quad \eta^{(1)} = -H + \lambda V_2, \quad \omega^{(1)} = 1 - \lambda,$$

$$\frac{d}{dt} = \frac{\partial}{\partial \tau} + \lambda \frac{\partial}{\partial \theta}.$$

These entirely determine the elements of the second order.

§ 9. As a first example we shall determine the system of inflexional circles for these displacements.

First we find

$$V_x = -2\lambda y; \quad V_y = 2H + 2\lambda x$$

$$J_x = 2H - 2\lambda^2 V_2 - 4\lambda^2 - 2\lambda' y$$

$$J_y = 2\frac{\partial H}{\partial \tau} + 4\lambda \frac{\partial H}{\partial \theta} + 2\lambda^2 \frac{\partial V_2}{\partial \tau} - 4\lambda^2 y + 2\lambda' x.$$

The equation of the inflexional circle, i.e. the locus of the points where the curve has an infinite radius of curvature, runs:

$$V_x J_y - V_y J_x = 0$$

or

$$2\lambda^3 (x^2+y^2) - (\lambda H + 2\lambda^2 H - \lambda^3 V_2)\, x -$$
$$- \left(\lambda \frac{\partial H}{\partial \tau} + 2\lambda^2 \frac{\partial H}{\partial \theta} + \lambda^3 \frac{\partial V_2}{\partial \tau} - \lambda' H\right) y - H(H - \lambda^2 V_2) = 0.$$

As λ is a constant, λ' a variable parameter, this gives:

For all displacements about the same pole of rotation the inflexional circles form a pencil; the base points lie on d; one of them is the pole of rotation, the other a point $H\left(\frac{H-\lambda^2 V_2}{\lambda^2}, 0\right)$. *In all these displacements H describes a point of inflexion.*

We found above: if $\mu(\xi, \eta)$ is the center of curvature of $M(x, y)$ we have

$$\xi = x - \frac{V_y}{V_x J_y - J_x V_y} V^2, \quad \eta = y + \frac{V_x}{V_x J_y - V_y J_x} V^2.$$

We substitute the values indicated for V_x, V_y, J_x and J_y but at the same time, in order to simplify the formulas, we choose the pole $\left(-\frac{H}{\lambda}, 0\right)$ of the movement as new origin, i.e. we put

$$\bar{x} = x + \frac{H}{\lambda}, \quad \bar{y} = y$$

$$\bar{\xi} = \xi - \frac{H}{\lambda} = \bar{x} - \frac{V_y}{V_x J_y - V_y J_x} V^2, \quad \bar{\eta} = \eta.$$

Thus we find

$$\bar{\xi} = \frac{(2B\bar{x} + 2C\bar{y} - \lambda' H\bar{y})\,\bar{x}}{8\lambda^3(\bar{x}^2+\bar{y}^2) + 2B\bar{x} + 2C\bar{y} - \lambda' H\bar{y}},$$

$$\bar{\eta} = \frac{(2B\bar{x} + 2C\bar{y} - \lambda' H\bar{y})\,\bar{y}}{8\lambda^3(\bar{x}^2+\bar{y}^2) + 2B\bar{x} + 2C\bar{y} - \lambda' H\bar{y}}$$

Here B and C are functions of λ, τ and θ, i.e. in all the considered elementary displacements — where the initial position and the value $\lambda = \frac{\partial \theta}{\partial \tau}$ have been chosen — they have the same values; λ' is a parameter that assumes any value.

If for the moment we choose also λ' constant, any point M has a definite center of curvature μ; in this case the aforesaid formulas express a well known quadratic correspondence between M and μ; it is especially interesting that in the inverse movement M is the center of curvature

of μ. Analytically this means that $\bar{x}$ and $\bar{y}$ are expressed in $\bar{\xi}$ and $\bar{\eta}$ through similar formulas.

The correspondence is quadratically involutory; to any line l described by M (or μ), there corresponds a conic as locus of μ (or M). The latter is sometimes called the conic of RIVALS, associated to l.

If now again we consider λ' as a parameter, it is at once evident [1] that:

The conics of RIVALS *that in the different displacements about the same pole of rotation correspond to the same line, form a pencil. The points of d — and these only — always have the same center of curvature.*

[1]) H. J. E. BETH, l.c.

Pathology. — *On Anophelism without malaria in the vicinity of Amsterdam* [1]. (2nd communication.) By N. H. SWELLENGREBEL, A. DE BUCK and E. SCHOUTE. (From the Institute for tropical hygiene, Department of the Royal Colonial Institute at Amsterdam.) (Communicated by Prof. W. SCHÜFFNER.)

(Communicated at the meeting of April 28, 1928).

In the vicinity of Amsterdam, including a "region I" where malaria occurs, and a "region II" where it is absent or rare, we described [2] two types of *A. maculipennis*: a long one (i.e. a longwinged form, bodylength and length of wing showing a high degree of correlation) and a short one, the last named being characterized moreover by a greater number of maxillary teeth. As a general rule the long type occurs in region II, the short one in region I. The difference between the two cannot be accounted for by external conditions (e.g. the salinity of the breedingplaces), because it persists after cultivation under exactly identical conditions. We therefore considered the two types as populations in which a short race or a long race dominate. We found no difference in their ability to act as a host for the parasite of simple tertian. The long form showed less appetite for human blood under experimental conditions. The difference of behaviour during hibernation proved to be the most important. Among the long forms hibernation begins in September; it is a complete one, i.e. hardly any blood is taken and there occurs a wide-spread and intensive development of the fat body. In the short forms semihibernation is the rule: feeding continues during the greater part of the winter, provided the temperature is not too low, and there is little development of the fat body. This behaviour of the long-winged population prevents its taking part in the transmission of malaria during the winter.

The necessity of proving the real existence of these differences by examining large numbers of mosquitoes and the fact that the behaviour during hibernation had only been established provisionally, made a reexamination desirable to check and extend our experience. The main facts resulting from this investigation are recorded here, for the details we refer to our extensive paper to be published later on.

Our present investigation includes 46 stations where we measured 11122 ♀ and established the percentage of females carrying blood, fat or eggs among 25065 ♀. Out of this material 4739 resp. 8945 ♀ came from the stations 4 (region I) and 29 (region II). We have regularly examined the Anopheline population of these two stations for a period of over two years. If nothing else could, this should convince us of the reality of the

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[1]) Part of this investigation was carried out with financial support from the malaria commission of the League of Nations.

[2]) These Proceedings **30**, p. 61.

MAP I

The numbers indicate the stations. The upper dotted line approximately indicates the southern limit of the area of endemic-, the lower one that of the area with sporadic malaria. The last one is the frontier between region I (North) and II (South).

I. The black sectors indicate the % (90° = 25 %) of Anophelines with a *wing above 134 units* (1 unit = 41.7 μ). The half circles to the right represent stable-mosquitoes, those to the left mosquitoes from shelters.

morphological difference of their populations. The results for each station separately may be gathered from the maps I and II; our general results are as follows:

	Winter 1926—'27:	Autumn 1927:
Stations in region II with pure longwinged population	wing: 131.4 ± 0.20; % fat: 58;	
Id. in reg. I with pure short-winged population	wing: 122.2 ± 0.14; „ „ 7;	wing: 119.2 ± 0.30; % fat: 7.5
Difference:	9.2 ± 0.25	
Stations, with segregation mainly in reg. II, "shelters"	wing: 129.2 ± 0.28; % fat: 45;	wing: 129,6 ± 0.32; % fat 66.5
Stations, with segregation mainly in reg. II, "stables"	„ ; 123.7 ± 0.30; „ „ : 21;	„ : 120.7 ± 0.31; „ „ 21.
Difference:	5.5 ± 0.41	Difference: 8.9 ± 0.42

Both periods:

			correlation length of wing and number of max. teeth
Station 4 (reg. I): wing: 121.5 ± 0.27; % fat: 6.5;		max. teeth: 17.9 ± 0.05;	+0.215 ± 0.034
Station 29 (reg. II): wing: 130.8 ± 0.27; „ „ : 79. ;		„ „ 17.1 ± 0.06;	+0.111 ± 0.038
Difference: 9.3 ± 0.39		Difference: 0.8 ± 0.08	

N.B. "Wing" = length of wing in units of 41.7 μ
"max. teeth" = number of maxillary teeth.

MAP II

II. Like map I but the sectors ($90° = 50\%$) indicate the *fat Anophelines.*

The measurement of Anophelines bred from larvae which had been cultivated under exactly identical conditions and only differed in the place the eggs came from (either region I or region II) confirmed last year's experience that the difference between the mosquitoes from these regions cannot be explained by external conditions. The result of the measurement of 514 ♀ and 398 ♂ from region II, 476 ♀ and 500 ♂ from region I is as follows:

region II.	wing ♀: 111.8 ± 0.32;	wing ♂: 102.5 ± 0.41;	max. teeth: 16.7 ± 0.04
region I.	„ „ : 108.8 ± 0.43;	„ „ : 93.4 ± 0.31;	„ „ 17.7 ± 0.05
Difference:	3.0 ± 0.54	9.1 ± 0.51	1.0 ± 0.07

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These results seem to confirm last year's. As a matter of fact there exists a notable difference. Our statement that the shortwinged race predominated in region I can be maintained (apart from a few exceptions) but its counterpart: predominance of the longwinged race in region II should not stand unmodified. For in this region a mixture of both races occurs almost everywhere [1]). This is brought to evidence in a very marked way during the winter by the occurrence of a *"segregation"* of the Anopheline population in two distinct groups: One, living in stables, is almost or quite identical with the short-winged type of region I; it continues to ingest blood and does not grow fat: it shows semihibernation. The other occurs in "shelters", i.e. uninhabited sheds, barns etc., also garrets and other appartments in human habitations; it is long-winged, fat [2]) (at least during the first two months of hibernation) and fasting. In most of the stations in region I there is no segregation and, consequently, very little difference between the mosquitoes from stables and shelters, both, as a rule, are short-winged, with little fat and taking blood under favourable conditions. In other words the short-winged population of region I is fairly pure, the long-winged one in region II is very mixed.

This is also born out by the positive correlation between the length of the wing and the number of maxillary teeth, which we failed to establish last year. This has changed by chosing our material as unmixed as possible. The purest material we have: laboratory-bred mosquitoes from stat. 4 in region I shows a correlation of $+0.445 \pm 0.040$. But similar mosquitoes from stat. 29 in reg. II do not show a higher figure than $+0.150 \pm 0.044$, whereas among Anophelines caught in nature in various stations of region II the correlation practically disappears ($+0.05 \pm 0.03$). Experimental or incidental mixing of long and short forms shows this (seeming) absence of correlation to be due to a gradual increase in the length of wing being accompanied at first by a corresponding rise in the number of teeth but afterwards by its decrease, an anomaly which can be eliminated by dividing the components of the mixed material according to their place of origin. This phenomenon is an important support of our view regarding the racial nature of our long and short types because, according to this view, the phenomenon is easily explained as a consequence of the short type having a larger number of teeth than the long one, whereas rejection of this standpoint leaves no room but for a highly artificial explanation.

Owing to the fact that the majority of wintering mosquitoes in region II are to be found in the "shelters" (i.e. long-winged ones) the average length of the wing in many stations is a high one. Stations with "medium-

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[1]) Last year there were a great number of stations not showing segregation, with long, fat and fasting mosquitoes in the stables. But many of them have turned to segregation in the autumn of 1927. This is why there is a blank under the heading "pure long-winged mosquitoes (autumn of 1927)" in the summary on p. 532. Moreover the difference between the mosquitoes from stables and shelters has increased.

[2]) There exists a fairly high positive correlation (0.42—0.52) between length of wing and fatness, among the mixed Anopheline population of the segregating stations. Among the pure population of region I it is insignificant (0.09—0.13). On the whole we take it to be a false correlation, a consequence of the general fatness during hibernation numbering among the racial characters of the long-winged type, but not of any direct correlative link between length of wing and fatness (as there exists e.g. between length of wing and number of maxillary teeth).

sized" Anophelines (as we called them last year) do no longer figure as harbouring a separate type of Anophelines but as segregating stations with a comparatively high number of stable mosquitoes (i.e. short ones). Consequently the distribution of the long- and short-winged type has changed very little indeed since the autumn of 1925—'26. But it has lost

MAP III

The sectors ($90° = 25\,\%$) indicate *long-winged Anophelines*, the white ones: wing of 130—134, black ones: wing above 134 units. The difference with map I is that in this case an average is established taking account of the mosquitoes in stables and shelters according to their numbers in each station. In this way the map can be compared with the one published in our 1st communication.

much of its value as an explanation of Anophelism without malaria, now we have become aware of the existence in region II of the short-winged type, which we must consider as the real vector of malaria, at least in winter.

There are two circumstances going a long way to help us out of this

difficulty, relating to *A. maculipennis* visiting human habitations and to local variations of its incidence.

Region II does not belong to those areas with Anopheles but without malaria where the mosquito is limited to the stables and does no longer enter houses. On the contrary, it is quite common there, in spring and autumn even more so than in the houses of region I. In summer it is less numerous in the houses of region II than of region I, but even then its relative incidence (compared to the numbers found in stables) is higher in the first [1]). A preliminary survey of the blood ingested by Anopheles in region I and II, by means of the precipitine-test, did not reveal any marked difference between the two with regard to their appetite for human blood [2]).

But the behaviour of Anophelines of region II visiting houses in winter is different from that in region I. The former belong to the "shelter"-mosquitoes, which carry on a complete hibernation (long-winged, fat, no blood); whereas the latter show semihibernation (short-winged, with little fat, blood).

Consequently the behaviour of the house-visiting mosquitoes in region I favours malarial infection in winter, whereas in region II it does not, whether there is segregation or not.

The preceding explanation does not account for malarial infection in summer. Even if the long-winged race should prove to be of little importance in this respect [3]) the presence of the short-winged race in region I can no longer be neglected. The local incidence of Anopheles, as shown in map IV, materially lessens this difficulty, because it is comparatively low in region II. The exceptions (e.g. stat. 52*a*, 52*b*, 58, 72) which would cause much trouble, if we wished to use the differences in the density of the Anopheline population as the sole and only explanation for the absence of malaria, do not raise any difficulty. For, although the Anophelines are numerous, they mainly belong to the long-winged type. This shows that the racial factor should not be neglected.

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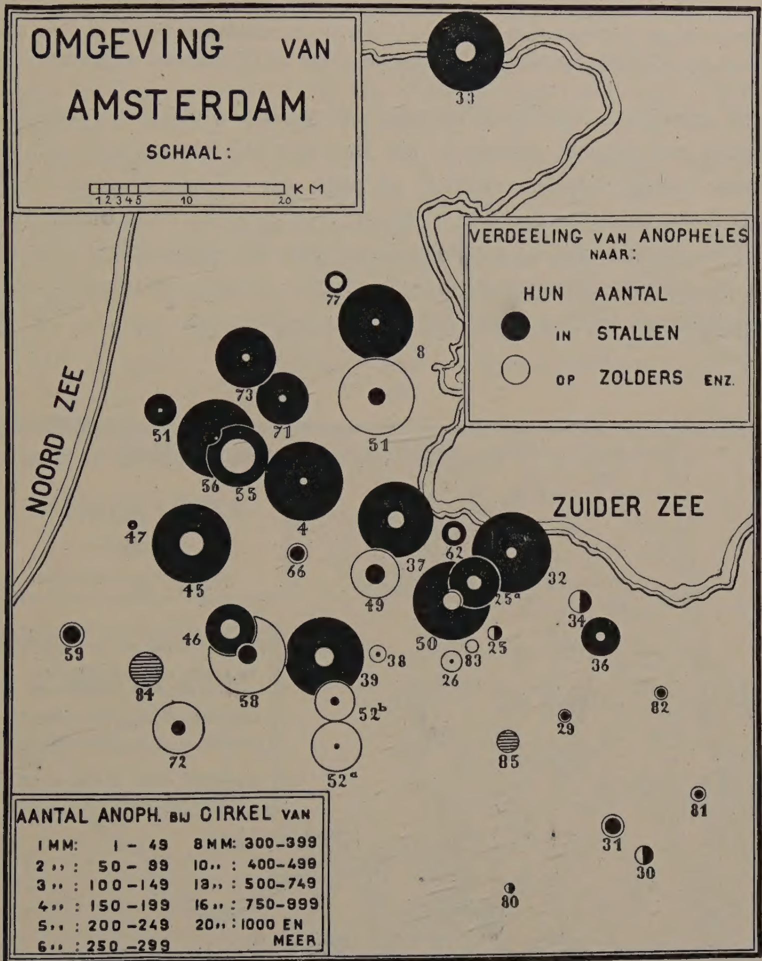
[1]) In spring and autumn 14—48 per house in region II, 12—18 in region I. In summer 1 per house, against 129 per stable in region II, and 5 against 1066 (1 : 213) in region I.

[2]) In the region without malaria 48 % of house-mosquitoes contained human blood against 14 % in the malarious region. But this requires confirmation by a more extensive investigation.

[3]) We are not nearly so sure about this as we are in the case of winter infection. Still the supposition is supported by the result of our feeding experiments with human blood which we have been carrying on for a period of 296 days with an average daily number of 80 ♀ from station 4 and 47 ♀ from stat. 29 in cages of $45 \times 30 \times 50$ cm^3. During all this time the long-winged type showed less appetite and in summer it suffered of a considerably higher mortality than the short-winged type. We do not wish to explain this difference by assuming a special "misanthropy" of the former but simply by its being evidently less able to stand confinement. This is of sufficient importance because the conditions required for *A. maculipennis* to act as an efficient malarial vector (as determined by JAMES *Brit. med. Jrl.*, Aug. 27, 1927), really amount to a close confinement like that in our cages.

There is the less cause for such a neglect as we have reason to believe that the low Anopheline incidence in region I is itself dependent on the

MAP IV



The circles, by the length of their radius, indi ate the largest *number of Anophelines* found in each station per stable (black) and per shelter (white), in Sept.—Oct. 1927. The circles are lying one on the other, the smaller one uppermost. When of equal diameter, two half circles take their place. The two striped circles indicate catches in July.

N.B. In region II the white circles indicate long-winged Anophelines, the black circles usually short-winged ones. In region I both circles indicate short-winged mosquitoes (save for a few exceptions).

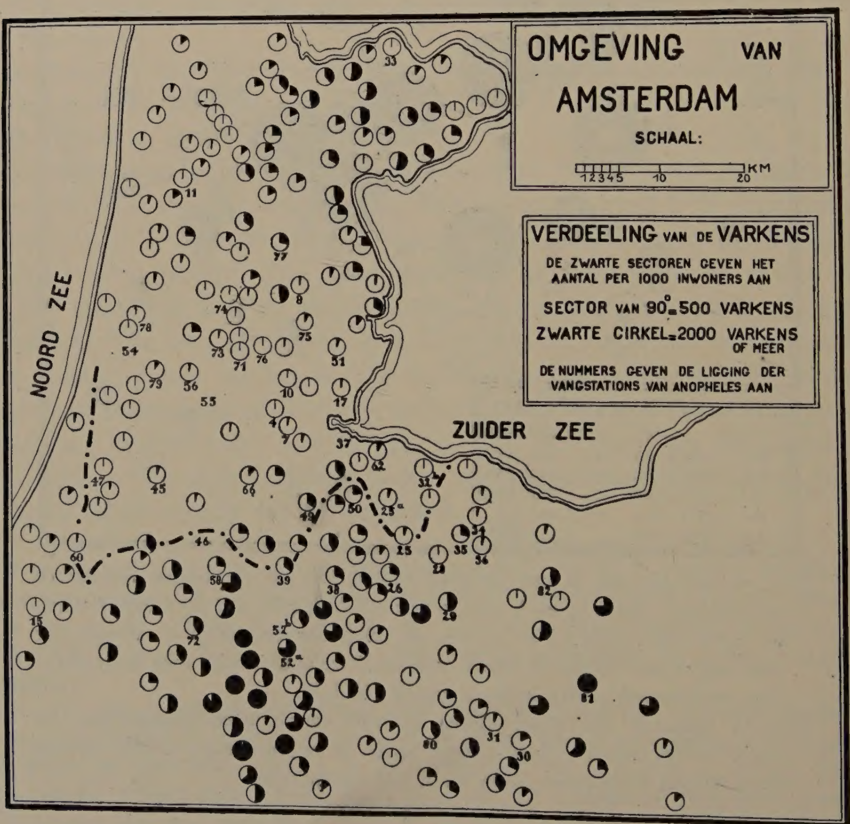
racial difference, by means of the larval incidence of the breedingplaces. which corresponds closely to the mosquito-incidence in its local variations, being low in region II and high in region I.

The low larval incidence in region II could not be explained by a general inferiority of its breedingplaces. The farthest our observations allowed us to go was to assume an

inhibition of the larval development in these breedingplaces affecting the shortwinged race inhabiting region II, as a consequence of the fresh water; our experience distinctly pointing to the favourable influence a certain degree of salinity exercices on these larvae in region I. A comparison of the food of the larvae in regions I and II revealed the former as less selective in the choice of their food; in summer they were not so much restricted to the use of green algae and flagellates [1]) as the latter, a circumstance which should affect their chances of survival and, consequently, the multitude of the race to which they belong.

There remains a notable difference between the regions I and II which we should not neglect although, for the moment, we cannot yet estimate its proper value, viz. the number of pigs, an animal of especial interest because the precipitine tests had shown it to provide more blood than any other. The distribution of pigs seems to open the perspective of a possible explanation of the absence of malaria in the frontier-stations of region II, with a numerous short-winged Anopheline population (e.g. No. 39, 46, 50).

MAP V



The black sectors (90° = 500 pigs) indicate the *numbers of pigs per 1000 inhabitants* in each municipality. The numbers show the situation of the stations mentioned in the maps I—IV.

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[1]) In accordance with VAN THIEL (*Bull. Soc. path. exot.* 1927. XX. 366) whose interpretation is somewhat different from ours.

A numerous porcine population might affect the local incidence of adult[1]) Anophelines by distributing the same number of adults over a greater number of stables. A comparison of the maps IV and V shows that the districts with a numerous porcine population are by no means confined to region II (North eastern portion of the province of N. Holland) but that there is a better agreement in the south.

Our last year's conclusion, explaining the absence of malaria in region II by the biological characteristics of our long-winged race which represents *A. maculipennis* in that region, is confirmed in so far as our present investigation supports the view that this race really exists and that its behaviour diminishes its importance as a malarial vector. But the phenomenon of segregation has shown us that the short-winged race likewise exists in region II. Still the investigation of the racial factor, although unable to explain, unaided, the Anophelism without malaria, has helped us to estimate at their proper value the local incidence of Anophelines (in general and in human habitations) and — provisionally — of pigs, as ancillary explanatory factors.

[1]) The larval incidence could not, of course, be influenced in this way.